String Consistency for Unified Model Building

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ABSTRACT

We explore the use of real fermionization as a test case for understanding how specific features of phenomenological interest in the low-energy effective superpotential are realized in exact solutions to heterotic superstring theory. We present pedagogic examples of models which realize SO(10) as a level two current algebra on the world-sheet, and discuss in general how higher level current algebras can be realized in the tensor product of simple constituent conformal field theories. We describe formal developments necessary to compute couplings in models built using real fermionization. This allows us to isolate cases of spin structures where the standard prescription for real fermionization may break down.

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1. Introduction

It is important in string theory to develop the dictionary that translates between four dimensional spacetime physics and the world-sheet properties of the string vacuum [1][2]. This will enable us to understand how specific phenomenological properties of possible interest in the low energy effective field theory are realized in superstring unification [3][4][5]. Much of the work to date in superstring phenomenology has focussed on the (N_R, N_L)=$(2, 2)$ compactifications [6] of the ten-dimensional $E_8 \times E_8'$ heterotic superstring [7]. The larger class of $(2, 0)$ vacua [8] has, however, remained largely unexplored except for the simplest abelian orbifold compactifications [9][10][11], a subset of which have an equivalent free fermionic realization [12][13].

In recent work [14], we have used *real fermionization*[1] [16][17][18] to understand how specific features of interest in the massless spectrum and tree-level couplings of the low-energy effective field theory are realized in exact solutions to string theory. Our starting point is the low-energy effective field theory. We will apply our knowledge of conformal field theory to find consistent ground states of string theory which embed spacetime features of possible phenomenological interest. Our preliminary results suggest many intriguing possibilities for phenomenology that are not present in either the $(2, 2)$ solutions or the known $(2, 0)$ orbifold compactifications. Some preliminary results have also been obtained by G. Cleaver [19]. L. Ibanez and collaborators [20] have recently begun a similar study of the phenomenological implications of higher level current algebras within the orbifold construction.

One of our goals is to make contact between string theory and more conventional field theoretic unification models. There are many indications that such a cross-fertilization of ideas would be fruitful. In the coming years the detailed exploration of the electroweak scale and the neutrino sector is likely to yield additional clues about short-distance physics besides the preliminary evidence for gauge coupling unification. In addition, increasingly accurate determinations of the parameters of the Standard Model will provide tighter constraints on unification schemes. The motivation for string theory is rooted in the successful unification of parity violating gauge interactions, quantum mechanics, and gravity [3][4][5]. It is therefore important to establish to what extent the low-energy particle

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[1] We use the expression "real fermionization" to distinguish this approach from free fermionic formulations [12][13][15] which assume a realization of the internal conformal field theory in either Weyl or Ising fermions, but have no *unpaired* Majorana-Weyl fermions.

physics consequences of string theory are robust. The string consistency conditions of modular invariance and world-sheet supersymmetry are extremely restrictive constraints on the spectrum. Thus we may expect guidance and insights for unification model builders by requiring string consistency of the effective field theory at the unification scale.

Supersymmetric grand unification models [21] suggest a picture in which radiative electroweak symmetry breaking and the large top quark mass are generated from a GUT-scale effective superpotential with a single third generation Yukawa coupling [22]. The distinct hierarchies in the pattern of fermion masses and mixings at the electroweak scale may be generated, in part, by higher dimension operators in the effective superpotential [23]. The recent results of Anderson, Dimopoulos, Hall, Raby, and Starkman [24] illustrate that the presence of a *small* number of higher dimension operators in the GUT-scale effective superpotential may be adequate to generate the observed masses and mixings. These higher dimension operators [24] are suppressed by powers of M_G over M_X, where $M_G \approx 10^{16}$ GeV, and M_X is another superheavy mass scale $\approx 10^{17}$ GeV. Restrictive flavor-sensitive selection rules are required in such scenarios to eliminate unwanted higher dimension operators and Yukawa couplings from the superpotential. Even more restrictive selection rules will be necessary in order to generate GUT scale masses for the triplet Higgs fields while keeping the supersymmetric Standard Model Higgs fields light [25][26]. Such restrictive symmetries appear unnatural from the point of view of an effective field theory. It is well-known in a general sense that string theory can provide such selection rules [27]. Less well-known is the possibility of using real fermionization to produce models which resemble conventional supersymmetric GUTs [17][28][18]. Real fermionization is also relevant to recent ideas about supersymmetric textures which do *not* invoke GUTs [29].

Finding explicit solutions to string theory that realize the required massless spectrum and selection rules of such mass matrix models will both provide guidance to model builders [30] and eventually give deeper insight into the origin of fermion masses and mixings. It should be noted that *unification* in the context of superstring theory has broader significance than the unification of the gauge couplings and (or) Yukawa couplings. The dynamical supersymmetry breaking sector, and a mechanism for feeding supersymmetry breaking to the low-energy matter, must be built into any consistent solution to string theory. Thus, string consistency is a powerful guiding principle in building *complete* supersymmetric models, which do not merely parametrize the weak scale effective Lagrangian but which also specify the origin of the soft supersymmetry breaking parameters.

Free fermionization is one of the oldest techniques known to string theorists and is the basis for the Ramond-Neveu-Schwarz formulation of the superstring [31][32][5]. The use of generalized GSO projections [32] to construct new solutions to string theory, given a consistent solution, was introduced in the context of the ten dimensional heterotic superstring in [33][34][35]. The ten dimensional ground states include a (non-supersymmetric) solution where the gauge symmetry is realized at level two [35]. In [12][13] this technique was applied to construct ground states with four dimensional Lorentz invariance. The fermionic formulation is based on the notion of current algebras and free fermionic representation theory [31][36][37]. A comprehensive discussion of non-renormalizable tree-level superpotential couplings can be found in [15]. Methods for analysing moduli dependence are given in [38][18][39], but these require further development.

A number of models of phenomenological interest have been constructed using free fermionization [40][41][42]. These models contain three generations of light chiral fermions and gauge groups like $SU(3) \times SU(2) \times U(1)$ or "flipped" $SU(5)$, realized by Weyl fermions on the world-sheet as current algebras at level one. The superpotential of the resulting low energy effective field theory has been computed for these models, using the techniques described in [15]. One then discovers interesting flavor-sensitive selection rules which restrict Yukawa couplings.

These realistic models belong to the subclass of free fermionic models which contain Weyl and Ising fermions, but do not contain any unpaired Majorana-Weyl fermions, which we call *real* fermions.[2] Models with only Weyl fermions produce simply-laced current algebras with Kac-Moody level one. This is because the local algebra of n Weyl fermions has central charge n and always contains n abelian currents. Models with both Weyl and Ising fermions can have reduced rank, because the Ising fermions soak up central charge without producing abelian currents. This also allows realizations of $SO(2n+1)$ at level one [43], and $SU(2)$ at level two [17].

Local algebras of $2n$ real fermions have central charge n and some number of abelian currents which is *variable* between zero and n. This richer set of local algebras allows us to realize current algebras which cannot be obtained in the subclass of models just discussed.

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[2] Properly speaking Ising fermions, which are right-left pairs of Majorana-Weyl fermions, are also real fermions. However it is very convenient for our analysis to let "real" denote only unpaired fermions, and identify Ising fermions separately. We hope that this usage does not cause confusion with respect to references [15] and [16], where "real fermions" includes Ising fermions.

In particular, real fermionization enables us to realize many current algebras at higher level. This in turn allows the appearance of adjoint Higgs in the massless spectrum, as needed for conventional GUT's [17]. Real fermionization also provides new embeddings of level one current algebras, and new possibilities for discrete symmetries in the effective field theory. We thus aim to exploit the techniques and successes of [40][41][42][15] while exploring a more general construction.

A non-trivial extension of these techniques is required when the underlying conformal field theory includes real fermions. The source of the difficulty is phase ambiguities in the explicit definition of the GSO projections and higher loop modular transformations for the real fermion conformal field theory. These phases play a crucial role in determining the massless spectrum and tree level couplings of the resulting models. A first attempt at resolving these ambiguities was made in [16]. We supplement that analysis by developing additional *tree-level* checks for string consistency.

The outline of this paper is as follows. In section 2 we review the well-known correspondence between gauge symmetry in spacetime and current algebras on the world-sheet [7]. This introduces the notion of world-sheet constraint algebras underlying the properties of the low energy effective field theory. In section 3 we explain in general how a higher level current algebra can be realized in the tensor product of *constituent* conformal field theories. We illustrate this with a toy model. Free fermion conformal field theories that embed both the gauge bosons and the chiral superfields transforming under such a current algebra, can be built into a consistent solution to string theory by using the real fermionization prescription of [16]. We explain how this works in the pedagogic discussion in section 4, presenting two examples with distinct fermionic embeddings of $SO(10)$. All of the results in this section were obtained with the use of a symbolic manipulation package developed by us [44]. In section 5 we address some of the formal developments necessary to understand real fermionization at a more fundamental level than the prescription of [16]. We use Verlinde's theorem [45] to relate the tree-level fusion algebra to the one-loop spin structure blocks in a way which allows unambiguous computation of the tree level correlators for real fermions. Combined with the methods of, e.g., [15], this will enable us to eventually automate the extraction of the tree-level superpotential. Our better understanding of real fermionization also allows us to probe cases of real fermion spin structures where the prescription of [16] breaks down. In the conclusion we make a critical appraisal of free fermionization, list some remaining problems, and discuss extensions of our methodology. We do not attempt to display any phenomenologically compelling models in this paper.

2. Spacetime symmetries and world-sheet operator algebras

The two-dimensional gauge principle of heterotic string theory is $(1,0)$ superconformal invariance [7][5]. In light-cone gauge,[3] the decoupling of timelike and longitudinal degrees of freedom results in a unitary conformal field theory, with a Hilbert space of positive norm. The field content includes the non-compact transverse spacetime coordinates, $X^\mu = \bar{X}^\mu(\bar{z}) + X^\mu(z)$, $\mu = 1,2$, and their Majorana-Weyl fermion superpartners, $\psi^\mu(\bar{z})$. In addition, there is an internal $(1,0)$ unitary conformal field theory of central charge $(9,22)$. Every physical state corresponds to the lower component of a conformal dimension $(h_R, h_L) = (\frac{1}{2}, 1)$ *world-sheet* superfield transforming under the $(1,0)$ superconformal constraint algebra.

The notion of finding world-sheet constraint algebras related to spacetime properties of the low-energy effective field theory was first explored in references [1][46]. We begin by reviewing the familiar example of gauge symmetry in order to explain how the constraint algebra can be used to build a solution to string theory embedding a specific low energy spectrum of fields.

In an $N=1$ spacetime supersymmetric vacuum all of the gauge symmetries are associated with the left-moving conformal field theory [7]. Then there must exist vertex operators of conformal dimension $(\frac{1}{2}, 1)$ which transform as spacetime vectors, corresponding to gauge bosons:

$$V^a(z, \bar{z}) \;=\; \zeta^\mu \psi^\mu(\bar{z}) J^a(z) e^{ik \cdot X} \qquad , \qquad (2.1)$$

where ζ^μ is the transverse polarization vector, $\zeta \cdot k = k \cdot k = 0$, and $J^a(z)$ is a dimension $(0,1)$ primary field in the left-moving internal conformal field theory. Gauge symmetry is therefore a consequence of an extension of the $(1,0)$ superconformal constraint algebra by dimension $(0,1)$ currents. The presence of the gauge bosons in the spectrum of massless fields implies that any chiral superfields that appear in the spectrum must satisfy the selection rules imposed by gauge invariance. In world-sheet language this implies strict agreement with the fusion rules of the world-sheet current algebra.

The operator product algebra of the dimension $(0,1)$ operators, $J^a(z)$, determines the structure constants and Schwinger term of a current algebra[4]:

$$J^a(z) J^b(w) \;=\; \frac{\delta^{ab}(k\psi^2/2)}{(z-w)^2} + \frac{if^{abc} J_c}{(z-w)} + \cdots \quad . \qquad (2.2)$$

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[3] We restrict ourselves to spacetime backgrounds with four dimensional Lorentz invariance.

[4] We will use the term current algebra for what is often referred to as an affine Kac-Moody algebra [47][48]. We will assume that the low energy gauge symmetry is related to a compact Lie group.

where ψ^2 is the length-squared of the highest root. This current algebra is, in general, based on the product of simple non-abelian and abelian group factors. The central charge from any simple group factor is given by the formula

$$c_k(G) \;\;=\;\; \frac{k \; dim(G)}{k + \tilde{h}} \qquad . \qquad\qquad (2.3)$$

The dual Coxeter number, $\tilde{h}$, is equal to C_A/ψ^2, where C_A is the quadratic Casimir of the adjoint representation. The *Kac-Moody level*, k, is restricted to take integer values due to the unitarity of the conformal field theory. It is common to normalize ψ^2 to 2 (or 1) so that the level coincides with (or is twice) the coefficient of the double pole term in (2.2). For our purposes it is more natural to normalize the coefficient of the double pole term to 1; the level is then read off from the norm of the roots.

In order to build a solution containing a specific low-energy spectrum of vector and chiral superfields, it suffices to find a realization of those gauge bosons which correspond to the *simple roots*, and the chiral superfields corresponding to the *highest weights* of the desired irreducible representations. The current algebra will automatically generate complete supermultiplets in the solution *if* care is taken to preserve the string consistency conditions of world sheet supersymmetry and modular invariance.

Thus, Lorentz invariance, spacetime supersymmetry and gauge invariance determine, in part, the emission vertex of any chiral superfield. Consider, for example, the vertex operator associated with a fixed helicity of a chiral superfield transforming as a spacetime fermion, $V_r^+(z, \bar{z})$. The vertex operator corresponding to the highest weight of an irreducible representation r will take the form,

$$V_r^+(z, \bar{z}) \;\;=\;\; S(\bar{z}) O(\bar{z}) f_r(z) F(z) e^{ik \cdot X} \qquad . \qquad\qquad (2.4)$$

We have left unspecified the dimension $(\frac{3}{8}, 0)$ primary field, $O(\bar{z})$, which must occur in the Ramond sector of the internal superconformal field theory; its form is restricted by the spacetime supersymmetry currents. $S(\bar{z})$, is a dimension $(\frac{1}{8}, 0)$ spin field in the Ramond sector of the conformal field theory of the Majorana-Weyl fermions $\psi^\mu(\bar{z})$. The Kac-Moody primary field $f_r(z)$ is of dimension $(0, h_r)$, and $F(z)$ is a gauge singlet of dimension $(0, 1 - h_r)$.

With higher level realizations of the current algebra, new matter representations can appear consistent with the requirement of unitarity of the underlying conformal field theory. This introduces new options for spacetime gauge and gravitational anomaly cancellation,

depending on which chiral fermion representations appear in the massless spectrum. A detailed tabulation of which representations and conformal dimensions are allowed in an affine Lie algebra at arbitrary level can be found in [49] and [48]. We should emphasize that, while unitarity is a restriction on which representations can appear at any given level, not every allowed representation need appear in a conformal field theory described by an asymmetric modular invariant.

3. Embedding higher level current algebras

The easiest way to realize a specific spacetime gauge symmetry in a consistent solution to string theory is to find an embedding of the current algebra in the tensor product of simple *constituent* conformal field theories. The best known constituents are free bosons and free fermions. However, as will become apparent, the method can be applied more generally.

The basic idea underlying the higher level current algebra realization is very simple. We begin by realizing the r abelian currents of the Cartan subalgebra of the group in a conformal field theory denoted as CFT_A. An abelian generator can always be realized by a chiral boson *with no loss of generality.* If we are realizing a non-abelian current algebra the chiral bosons have rational conformal dimensions (see, for example, [48][50]). Thus CFT_A is constructed using r chiral bosons with conformal dimensions, $h_L=p_L^2/2=m/n$, with m, n integers.

For a higher level realization it is not possible to construct the remaining currents of the non-abelian current algebra using only operators of the free boson conformal field theory, CFT_A.[5] Thus what we actually need is a tensor product of CFT_A with some other constituents, which we will denote collectively as CFT_B. In this paper we restrict ourselves to the cases where CFT_B is constructed using unpaired Majorana-Weyl (real) fermions. This is a *strong* restriction on which gauge groups and representations can be obtained in this class of solutions. The obvious generalization is to allow as constituents of CFT_B any of the unitary conformal field theories with central charge $c<1$ [52]. These conformal field theories have a finite number of chiral primaries under the Virasoro algebra and rational conformal dimensions, $h_i<1$. They have no spin one currents. The corresponding Virasoro characters, which enter the string partition function, have well-defined modular transformation properties.

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[5] Higher level realizations using *twisted* free bosons are possible: see [51].

If the tensor product $CFT_A \times CFT_B$ successfully realizes a current algebra, then the total central charge $c_A + c_B$ must *at least* equal $c_k(G)$. If $c_A + c_B > c_k(G)$ this implies that we have realized, in addition to the higher level current algebra, some *other* holomorphic algebra which contains no currents. We will refer to this other algebra as a *discrete holomorphic operator algebra.*

Thus the (left-moving) stress tensor for a higher level current algebra realization has, in general, two distinct decompositions:

$$
\begin{aligned}
T &= T_A + T_B \\
&= T_{KM} + T_{discrete} \quad ,
\end{aligned}
\tag{3.1}
$$

where T_{KM} denotes the Sugawara form of the stress tensor of the higher level current algebra, and $T_{discrete}$ denotes the coset algebra formally defined by the relation (3.1).

Two observations of considerable practical importance are as follows. The rank of the low-energy gauge symmetry in a four dimensional ground state is bounded by the central charge of the left-moving internal conformal field theory, $\sum_i rank(G_i) \leq 22$. Also, the dimensions of individual matter representations that can appear at the massless level are bounded by the condition, $\sum_i h_L^i \leq 1$ [53][11].

The conformal field theory of a chiral boson, $\phi(z)$, with rational-valued momentum, p, is equivalent to that of a Weyl fermion, $\lambda(z)$, with fermionic charge, Q:

$$
\begin{aligned}
\partial \phi &\rightarrow\ : \lambda^\dagger \lambda : \\
\hat{p} &\rightarrow\ \widehat{Q} = \widehat{F} - \frac{v}{n}\mathbf{1} \quad .
\end{aligned}
\tag{3.2}
$$

Here $\widehat{F}$ is the fermion number operator, and the vacuum fermionic charge, v/n, is rational-valued. The abelian current is realized by the Weyl fermion bilinear. Fermionic representations of current algebras that utilize fermion bilinears are well-known. The non-simply-laced algebras at level one can be realized by Majorana-Weyl fermions. For example, the generators of $SO(2n + 1)$ are realized by n Weyl fermions and a single Majorana-Weyl fermion, or equivalently, $2n + 1$ Majorana-Weyl fermions [43]. The currents are the $2n(2n + 1)/2$ Majorana-Weyl fermion bilinear pairs.

When we realize the Cartan currents using Weyl fermion bilinears, every distinct group weight will be realized as a unique set of fermionic charges. This representation of weights in a basis defined by fermionic charges is fixed once we specify the fermionic charges of the r simple roots [54]. We then identify in CFT_A holomorphic operators, $\phi_{q_1,...q_r}^a(z)$ with the

correct fermionic charges $(q_1, \ldots q_r)$ to represent all of the currents, $J^a(z)$, of the higher level algebra. Since these primaries may not have conformal dimension 1, we then must identify other operators in CFT_B to make up the difference. Thus

$$J^a(z) = \phi^a_{q_1, \cdots q_r}(z) \times \phi^a_B(z) \qquad . \tag{3.3}$$

The above also holds for chiral bosons when we map weights into momenta.

3.1. Canonical Embeddings

Let us explain, from first principles, how one can identify a realization of some given current algebra at arbitrary level, assuming explicit knowledge of the conformal dimensions, operator product coefficients, and Virasoro characters of the chiral primaries of the constituent conformal field theories.

There are many possible free field embeddings of any given current algebra. We will refer to the embedding with the lowest possible total conformal anomaly as the *canonical* embedding. One advantage of using a canonical embedding of the roots (e.g., the standard Cartan-Weyl basis for a level one realization) is that the model builder avoids the pitfall of unexpected extra gauge symmetry such as U(1) factors in the final solution.

We begin with a realization of the Cartan subalgebra of the group. Each of the r abelian currents is realized by a chiral boson

$$h_i \;=\; \partial \phi_i \qquad i \;=\; 1, \cdots, r \quad , \tag{3.4}$$

where r is the rank of the gauge group. These are operators of conformal dimension one. Let us assume that the momenta of the individual chiral bosons are quantized such that

$$\phi_i(\sigma_1 + 2\pi, \sigma_2) \;=\; \phi_i(\sigma_1, \sigma_2) \;+\; 2\pi p_i \quad . \tag{3.5}$$

Consider vertex operators of non-zero momentum

$$V_j^{(\pm)} \;=\; C_j(\hat{p}) : e^{\pm i \mathbf{p}_j \cdot \phi} : \quad , \tag{3.6}$$

where $\mathbf{p}_j$ and ϕ are r dimensional vectors, and the $C_j(\hat{p})$ are cocycle operators. This is the familiar vertex operator construction used in the $E_8 \times E_8$ heterotic string [5]: if the $\mathbf{p}_j$ lie on the root-lattice of a simply-laced group the commutation relations of the vertex operators, with cocycle operators appropriately defined, will reproduce the structure constants of the

associated current algebra. The normalization of the abelian currents is not fixed until we specify the realization of the nonzero roots.

Now consider a specific example of this construction in the context of heterotic string theory. Begin with five copies of the root lattice of $SU(2)$

$$([\pm\sqrt{2}, 0, 0, 0, 0]) \qquad , \qquad (3.7)$$

where the square brackets denote permutations, and we have normalized the roots to length $\alpha^2=2$. Let us assume that this lattice is embedded in the 22 dimensional sublattice of an even self-dual Lorentzian lattice of dimension $(6, 22)$ [55][5]. The states corresponding to the roots of $(SU(2))^5$ given in (3.7) will then appear at the massless level, with $p_L^2=2$, $h_L=1$, and correspond to spacetime gauge bosons. The realization of the gauge symmetry is at level one. From the properties of self-dual lattices, it follows that the weight lattices of $(SU(2))^5$

$$([\pm\tfrac{1}{\sqrt{2}}, 0, 0, 0, 0]) \qquad (3.8)$$

are present in the $(6, 22)$ dimensional lattice [50]. Ignoring the precise constraints from modular invariance, imagine that we perform a sequence of orbifold twists accompanied by shift vectors embedded in the $(SU(2))^5$ lattice whose net effect is to project out the *individual* roots and weights but leave intact the lattice points

$$([\pm\tfrac{1}{\sqrt{2}}, \pm\tfrac{1}{\sqrt{2}}, 0, 0, 0]) \qquad , \qquad (3.9)$$

where all permutations are included. The *counting* of states is correct to fill out the adjoint representation of the group $SO(10)$, $5 \cdot 4 \cdot 2 \; + \; 5$ giving a total of 45 states if we include the states corresponding to the five abelian currents.

Suppose we rescale the normalization of the abelian currents by a factor of two. Then the length of the lattice vectors in (3.9) is exactly what is needed for a level two realization of the gauge symmetry. The only problem is that the states of non-zero momentum no longer appear at the *massless* level because the (left) conformal dimension is only $\frac{1}{2} \cdot p_L^2 = \frac{1}{2}$. This problem is easily fixed. The central charge of $SO(10)$ at level 2 can be read off from the formula (2.3) given in the previous section, where $C_A=2(2n-2)$ for $SO(2n)$. The central charge of the embedding conformal field theory of five chiral bosons is $c=5$. Thus, if we can find a (rational) conformal field theory with central charge $c>4$, primary fields of conformal dimension $\frac{1}{2}$, and *no* dimension one currents, by tensoring together the two conformal field theories it should be possible to find an embedding of these states at the

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massless level. A necessary requirement is that we exactly match the conformal dimensions and counting of states given above *without* modifying their fusion rules.

Let us outline how to find such an embedding for our toy model.[6] The first five left-moving entries of the $(6, 22)$ dimensional lattice before twisting have already been determined (3.7), (3.8). Let us assume that the next eight entries embed the root-lattice of $SO(16)$

$$([\pm 1, \pm 1, 0, 0, 0, 0, 0, 0]) \qquad . \tag{3.10}$$

Together with the spinor and conjugate spinor weights of $SO(16)$,

$$(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}) \qquad , \tag{3.11}$$

one obtains the E_8 lattice. This lattice is easily embedded in an even self-dual Lorentzian lattice given by the sum of the root and weight lattices of $(SU(2)_L)^6 \times (SU(2)_R)^6 \times E_8 \times E_8'$ [55]. The self-dual lattice describes the compactification of the ten dimensional $E_8 \times E_8'$ heterotic string on an $(SU(2))^6$ torus.

The conformal field theory underlying the E_8 lattice has a fermionic representation [7][5]. The eight chiral bosons can be fermionized as follows:

$$\begin{aligned}
\partial \phi_i &\rightarrow\ : \lambda_i^\dagger \lambda_i : &&i = 7, \cdots, 14 \\
e^{\phi_i} &\rightarrow\ (-1)^{\widehat{F}_i} \lambda_i \\
\hat{p}_i &\rightarrow\ \widehat{F}_i\ -\ \frac{v_i}{2} \mathbf{1} &&.
\end{aligned} \tag{3.12}$$

The equivalence between momentum and fermionic charge for momentum quantized in half-integer units, $p_i = n/2$, implies that the conformal field theory of the Weyl fermions has two sectors. The two sectors correspond to choosing Neveu-Schwarz (antiperiodic) or Ramond (periodic) boundary conditions for the fermions, respectively, $v_i = 0, 1$:

$$\lambda_i(\sigma_1 + 2\pi, \sigma_2)\ =\ -e^{\pi i v_i} \lambda_i(\sigma_1, \sigma_2) \qquad . \tag{3.13}$$

The roots of $SO(16)$ correspond to oscillator excitations in the Neveu-Schwarz sector. The spinor weights given in (3.11) correspond to states in the Ramond sector, with $F_i = 0, 1$,

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[6] The reader will recognize an obvious parallel with the asymmetric orbifold construction in the discussion that follows.

and $v_i=1$, for all i. In the one-loop vacuum amplitude this sector is labelled by a vector specifying the boundary conditions of the individual fermions, v_i, $i = 1, \cdots, 8$,

$$(1\ 1\ 1\ 1\ 1\ 1\ 1\ 1) \quad . \tag{3.14}$$

Thus, in the absence of constraints from any other sectors, this sector contributes the 2^8 spinor and conjugate spinor weights of $SO(16)$ in the one-loop vacuum amplitude.

For convenience, we can rewrite the Weyl (complex) fermions as Majorana-Weyl fermions, $\lambda_i = \psi_i^{(1)} + i\psi_i^{(2)}$. The two Majorana-Weyl fermions associated with each of the eight Weyl fermions share the same boundary condition in every sector summed over in the one-loop vacuum amplitude. Implicitly, we are now allowing for the possibility of Majorana-Weyl fermions which are no longer paired into complex fermions. Some of these may be right-left paired into Majorana (Ising) fermions. Any Majorana-Weyl fermions which are truely unpaired we call *real* fermions. In the absence of a complexification of the Majorana-Weyl fermions, a conserved fermionic charge, or equivalently, a conserved bosonic momentum, can no longer be defined. We can re-label the sector (3.14) contributing the spinor weights of $SO(16)$ by the corresponding boundary condition vector ($v_i=1$, $i = 1, \cdots 16$) for *sixteen* real fermions:

$$(11\ 11\ 11\ 11\ 11\ 11\ 11\ 11) \quad . \tag{3.15}$$

Ignoring once again the constraints from modular invariance, consider the possibility of blocks of *chiral Z_2* twists on the E_8 lattice accompanied by the shift vectors embedded in the $(SU(2))^5$ lattice such that all of the E_8 gauge symmetry is broken to a discrete subgroup. This corresponds to introducing new sectors in the one-loop vacuum amplitude which contribute states of non-zero momentum in the conformal field theory of the chiral bosons, ϕ_i, $i=1, \cdots, 5$, corresponding to the lattice points (3.9), matched with the tensor product of Ramond ground states for blocks of eight real fermions chosen from the set, $\psi_i^{(j)}$, $i=1, \cdots, 8$, and $j = 1, 2$. In order to break all of the E_8 gauge symmetry we need to include at least four sectors in the one-loop vacuum amplitude, corresponding to the following boundary condition vectors for the sixteen real fermions:

$$
\begin{aligned}
&(1111\ 1111\ 0000\ 0000) \\
&(0000\ 1111\ 1111\ 0000) \\
&(1100\ 1100\ 1100\ 1100) \\
&(1010\ 1010\ 1010\ 1010) \quad .
\end{aligned}
\tag{3.16}
$$

The contribution to the left conformal dimension from the Ramond vacuum energy in each of these sectors is $\frac{1}{16} \cdot 8 = \frac{1}{2}$. Therefore, oscillator excitations described by fermion bilinears of the form, $: \psi_j \psi_k :$, contribute with conformal dimension *greater* than one in these sectors and are pushed up to the massive level. The sectors (3.16) also act as constraints on the *untwisted* sector, i.e., the sector with all fermions in the Neveu-Schwarz vacuum, so that these dimension one states are projected out of the spectrum by the requirement of modular invariance. Thus the untwisted sector does not contain any currents. Of course, one must still be concerned with additional dimension one states that can contribute from twisted sectors. Choosing the projections on the spectrum such that no additional dimension one currents appear requires a detailed knowledge of the constraints from one-loop modular invariance. While this certainly could be done, we will not pursue this toy model any further. Certain elements of the toy model can, however, be recognized in the examples of section 4.

The embedding (3.9) of the roots of $SO(10)$ in the doublets of five copies of $SU(2)$ is a special case of the embedding of the roots of $SO(2n)$ at level $k=2$ in the fundamental weight-lattices of the group $(SU(2))^n$. The pattern further generalizes to an embedding of the roots of $SO(2n)$ at level k in the momentum lattice of n chiral bosons, with momentum quantized in units of $1/\sqrt{k}$. Embeddings of the roots of the special unitary groups can be worked out by the same method.

3.2. Fermionic Embeddings

Now let us specialize to the case where the $c=1$ constituents of CFT_A are Weyl fermions and the constituents of CFT_B are $c=\frac{1}{2}$ Majorana-Weyl fermions.

It is important to distinguish between a fermionic *embedding* and a fermionic *representation* of a current algebra. A fermionic embedding is simply a mapping of the roots of a Lie algebra into fermionic charges. A fermionic representation is an embedding where the total conformal anomaly of the fermions equals the central charge of the Kac-Moody algebra. An example of a higher level fermionic representation is $SU(2)$ at level two realized by three Majorana-Weyl fermions.

Fermionic representations may or may not exist depending on the group and the level of the current algebra. The orthogonal groups at level one have fermionic representations. But the special unitary groups at level one are only obtained in the fermionic embedding of the group $SU(n) \times U(1)$. The 'extra' U(1) in a fermionic embedding cannot be broken by

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standard stringy symmetry breaking techniques, e.g., a Z_2 twist, without simultaneously breaking the nonabelian symmetry.

These statements have counterparts for fermionic realizations of higher level current algebras. A fermionic embedding determines the level of the current algebra by fixing the lengths-squared of the nonzero roots. To be precise, let $\overrightarrow{Q}=(q_1, q_2, \ldots q_n)$ denote the fermionic charges of a root; then $\overrightarrow{Q}^2$ must have the same value for all the roots (all the long roots if the group is not simply laced). The level is then given by [17]:

$$k = \frac{2}{\overrightarrow{Q}^2} \qquad . \tag{3.17}$$

An example of a higher level fermionic embedding is the minimal *fermionic* embedding of the roots of $SO(10)$ at level two, which requires *six* Weyl fermions [17](see section (4.1)). Since there is an additional abelian generator orthogonal to the space spanned by these roots, the six Weyl fermions actually provide an embedding of $SO(10) \times U(1)$. It is also possible to find fermionic embeddings of special unitary groups within a semi-simple group: for example $SU(5) \times SU(2)$, with the $SU(5)$ at level two and the $SU(2)$ at level four, and $Sp(6) \times SU(3)$, with the $Sp(6)$ at level one and the $SU(3)$ at level two.

A fermionic realization is a fermionic embedding or representation together with a realization of the currents and physical states corresponding to the gauge bosons in a consistent string vacuum. A fermionic embedding does not necessarily extend to a fermionic realization, since we are restricting the constituents of CFT_B to be real fermions. A necessary condition is that one can identify dimension (0,1) operators with fermionic charges corresponding to all the roots. For the types of operators in CFT_A which are relevant for constructing currents, there is a simple relation between their fermionic charges and their conformal dimension[12]:

$$h = \frac{1}{2}\overrightarrow{Q}^2 \qquad . \tag{3.18}$$

Simple examples are single Neveu-Schwarz fermion operators ψ, $\psi^\dagger$, (which create single fermionic excitations of the Neveu-Schwarz vacuum) having $h=1/2$ and fermionic charge ± 1, and single Weyl fermion *twist* fields σ, μ, (which create the doubly-degenerate Ramond vacua from the Neveu-Schwarz vacuum) having $h=1/8$ and fermionic charge $\pm 1/2$.

As will be discussed further in section 5, the $c=1/2$ conformal field theory of a single Majorana-Weyl fermion contains primary fields with conformal dimension 0 (the identity), 1/16 (twist fields), or 1/2 (the Neveu-Schwarz fermion). Thus there are a limited number

of ways to construct currents. In particular, if $\overrightarrow{Q}$ represents the fermionic charges corresponding to some root, then the current corresponding to that root exists only if there is a solution to

$$1 = \frac{1}{2}\overrightarrow{Q}^2 + \left(\frac{m_1}{16} + \frac{m_2}{2}\right) \tag{3.19}$$

where m_1, m_2 are nonnegative integers.

Combining (3.19) with (3.17), we obtain an important restriction[7] on the possible levels for current algebras with fermionic realizations:

$$k = 1, \ 2, \ 4, \ 8, \ \text{or } 16 \qquad . \tag{3.20}$$

It should be noted that the higher level fermionic embedding does *not* uniquely determine the fermionic realization of the current algebra. An example is given in the next section.

4. Real Fermionization: examples

To understand in detail how the constraints from modular invariance determine the spectrum and couplings of a solution, it is useful to focus on a specific set of constituent conformal field theories. Fermionization of the internal $(2,0)$ unitary conformal field theory is a relatively straightforward technique for generating explicit solutions to the string consistency conditions [12][13][16]. In this section we will explain how the ideas we have introduced in the previous two sections get implemented in the context of specific examples. These examples have been constructed to illustrate how particular phenomenological aspects find their realization in string theory. Although our methodology has the potential of steadily leading to more phenomenologically compelling models, the models discussed here were selected for their pedagogic value only.

The constituent fields of the internal superconformal field theory are a collection of Majorana-Weyl fermions. Some number of these are paired into right-moving or left-moving Weyl fermions, or into right-left paired Majorana (Ising) fermions. The total central charge sums to $(9, 22)$ for a heterotic vacuum with four dimensional Lorentz invariance.[8] Including the two right-moving Majorana-Weyl fermions with a spacetime index gives a total of 20 right-moving and 44 left-moving constituent fermions.

---

[7] Condition (i) of section 5.3 rules out the case $k=16$.

[8] "Heterotic" refers to the construction of the four dimensional solutions; it is not necessarily the case that these solutions possess a large-radius limit which recovers the ten dimensional heterotic superstring.

The boundary conditions of the fermions about the two non-contractible loops on the torus specifies their spin-structure. Consider first the Weyl fermions which are obtained by a complexification of a pair of Majorana-Weyl fermions, $\lambda(z) = \psi_1(z) + i\psi_2(z)$. The fermionic charge (bosonic momentum) is allowed to take any rational value. The possible (twisted) boundary conditions are denoted:

$$
\begin{aligned}
\lambda(\sigma_1 + 2\pi, \sigma_2) &= -\mathrm{e}^{\pi i v}\, \lambda(\sigma_1, \sigma_2) \\
\lambda^\dagger(\sigma_1 + 2\pi, \sigma_2) &= -\mathrm{e}^{-\pi i v}\, \lambda^\dagger(\sigma_1, \sigma_2) \qquad ,
\end{aligned}
\tag{4.1}
$$

where v takes any rational value restricted to the domain $-1 < v \leq 1$. The boundary conditions described by eq. (4.1) reduce to a possible sign flip for both Majorana-Weyl fermions combined with a rotation of the Majorana-Weyl fermions among themselves:

$$
\begin{pmatrix} \psi_1(\sigma_1 + 2\pi, \sigma_2) \\ \psi_2(\sigma_1 + 2\pi, \sigma_2) \end{pmatrix} = - \begin{pmatrix} \cos(\pi v) & \sin(\pi v) \\ -\sin(\pi v) & \cos(\pi v) \end{pmatrix} \begin{pmatrix} \psi_1(\sigma_1, \sigma_2) \\ \psi_2(\sigma_1, \sigma_2) \end{pmatrix} .
\tag{4.2}
$$

A right-moving and a left-moving Majorana-Weyl fermion paired to form a Majorana (Ising) fermion are both either periodic (Ramond) or antiperiodic (Neveu-Schwarz) in every sector of the partition function. Any Majorana-Weyl fermions which are unpaired are called *real* fermions. Real fermions take Ramond or Neveu-Schwarz boundary conditions.

In general, the one-loop vacuum amplitude (partition function) $\mathscr{Z}_{\text{Fermion}}$ can be written as a sum over all possible spin structures generated from a set of *basis vectors*, $\{V_i\}$, i.e., the boundary condition vectors for the constituent fermions which span the sectors summed over in the partition function:

$$
\mathscr{Z}_{\text{Fermion}}(\tau) = \sum_{\alpha, \beta} C^{\alpha V}_{\beta V} \mathscr{Z}^{\alpha V}_{\beta V}(\tau) ,
\tag{4.3}
$$

where $\{\alpha_i\}$, $\{\beta_i\}$ are independent sets of nonnegative integers both generating linear combinations of the basis vectors vectors V_i. The $C^{\alpha V}_{\beta V}$ are projection coefficients associated with each specification of spin structure; they determine the phase with which the states in a particular sector contribute to the partition function.

The $\mathscr{Z}^U_V(\tau)$ for each spin structure are defined in a Hamiltonian representation as:

$$
\mathscr{Z}^U_V(\tau) = \mathrm{Tr}\left\{ (-1)^{\mathbf{U} \cdot \widehat{\mathbf{F}}_{\mathbf{V}}} \exp\left(2\pi i \tau \widehat{H}^L_{\mathbf{V}} - 2\pi i \bar\tau \widehat{H}^R_{\mathbf{V}} \right) \right\} .
\tag{4.4}
$$

For the Weyl and Ising components, the GSO projection operator, $(-1)^{\mathbf{U} \cdot \widehat{\mathbf{F}}_{\mathbf{V}}}$, is defined in the obvious way from the fermion number operator $\widehat{\mathbf{F}}$; for real fermions its explicit form is more complicated [16].

The coefficients $C_{\beta V}^{\alpha V}$ are conveniently rewritten as:

$$C_{\beta V}^{\alpha V} = e^{2\pi i[-\alpha_i k_{ij}\beta_j + \alpha_i s_i - \beta_i s_i]} \qquad , \qquad (4.5)$$

where the k_{ij} are rational parameters, repeated indices are summed, and s_i takes values 0 or $-1/2$, depending on whether the basis vector V_i contributes spacetime bosons or fermions, respectively. To define a solution, it is only necessary to specify k_{00} and the k_{ij} for $i>j$; the other k_{ij} are then fixed by modular invariance.

A solution takes the form of a definite *spectrum* of physical states that survive all of the *projections* imposed by string consistency. The partition functions for interesting solutions sum over thousands of spin structures, thus it is clearly not practical to perform the required projections by hand. Instead we have developed a symbolic manipulation package [44] which automatically extracts the massless spectrum of solutions compatible with the fermionic formulation introduced by Kawai, Lewellen, Schwartz, and Tye (KLST)[16]. This program takes as input a list of basis vectors, V_i, and projection coefficients, k_{ij}. It then checks for string consistency, performs the GSO projections, checks for spacetime supersymmetry, identifies the gauge group and its embedding from the gauge bosons in the massless spectrum, then outputs the full massless spectrum organized into irreps of the gauge group. The tree couplings of physical states can be inferred from their decomposition into primary fields of the constituent conformal field theories. However, because of the new formalism required for real fermions (as will be described in the next section) we have not yet automated the extraction of the full tree-level superpotential.

The notion of embeddings makes such a methodology particularly well-suited to realizing operator algebras that determine specific spacetime symmetries. Every model contains the untwisted (i.e. all Neveu-Schwarz) sector, which ordinarily would contribute the gauge bosons of the group $SO(44)$, or its regular subgroups. In the solutions we are interested in, most of the gauge bosons and chiral matter do *not* appear in the untwisted sector. Rather, the twisted sectors embed most of the gauge bosons and the matter representations. This is an important distinction from the familiar $(2,2)$ compactifications, or $(2,0)$ constructions that are related to $(2,2)$ compactifications [11][56], where the low-energy gauge symmetry is realized in the untwisted sector.

The spin structures are specified by listing the basis vectors V_i, which have 20 right-moving and 44 left-moving components separated by a double vertical line. Since we use a 64 component Majorana-Weyl notation, Weyl fermion spin structures are written as left-left or right-right pairs, and Ising fermion spin structures by left-right pairs. As always

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0,1 denotes Neveu-Schwarz/Ramond boundary conditions; we also use $++$ and $--$ to denote a Weyl fermion whose boundary condition is $\mp i$ times itself when taken around a noncontractible loop.

The first two components of every vector refer to the right-moving fermions with spacetime indices, $\psi^\mu(\bar{z})$. Thus (00) in these slots indicates a spacetime boson; if $\psi^\mu(\bar{z})$, $X^\mu(\bar{z})$, and $X^\mu(z)$ are not excited the resulting massless states in such a sector are scalars. On the other hand, (11) indicates a spacetime fermion, in this case the two possible values of the "fermionic charge", $\pm 1/2$, distinguish the two helicity states.

4.1. Model A

This example has $N{=}1$ spacetime supersymmetry, $SO(10)$ realized at level two, chiral fermions, and Higgs in the 10 and 45 of $SO(10)$.

V_0: $(11111111111111111111\,\|\,11111111111|11111111111111|111|11111111111)$
V_1: $(11100100100100100100\,\|\,00000000000|00000000000000|000|00000000000)$
V_2: $(00000000000000000000\,\|\,11111110000|11111110000000|000|00000000000)$
V_3: $(00000000000000000000\,\|\,00000000000|00001111111000|000|00000000000)$
V_4: $(00000000000000000000\,\|\,11000011111|11001100110110|000|00000000000)$
V_5: $(11100100010010010010\,\|\,11110000110|10101010101010|010|11000000000)$
V_6: $(11010010100100001001\,\|\,11110000110|10100101101010|101|00000000000)$
V_7: $(11001001001001100100\,\|\,11110000110|11110000111111000|000|00110000000)$
V_8: $(00110110110110000000\,\|\,00000000000|01010101010111|000|00000000000)$
V_9: $(00000000000000000011\,\|\,00000000000|00000000000000|011|001100++++++++)$

<div align="center">Model A</div>

The k_{ij} for $i > j$ and k_{00} are all zero except for the following which are equal to $-1/2$: k_{71}, k_{73}, k_{81}, k_{83}, k_{85}, and k_{86}.

Apart from the spacetime fermions, the right-movers in this model correspond to 7 world-sheet Weyl fermions and 4 Majorana-Weyl fermions. Three of the Majorana-Weyl fermions (in slots 17, 19, 20) pair up with left-movers to make 3 Ising fermions; the fourth Majorana-Weyl fermion (in slot 16) is associated with 15 left-moving Majorana-Weyl fermions as a block of 16 real fermions. There are 7 fermionic charges associated with the complex right-movers; they take values 0, $\pm 1/2$, and ± 1 for massless states. These charges result in discrete symmetries in the low-energy effective theory.

The left movers are separated into four blocks, embedding the visible matter gauge quantum numbers, the real fermion spin structures, the Ising fermion spin structures, and the hidden sector gauge quantum numbers. In this example the first 12 left-mover slots denote 6 Weyl fermions. The 6 associated fermionic charges take values 0, $\pm 1/2$, and ± 1 for massless states; these charges are simply weights of the visible gauge group $SO(10)\times U(1)$, in the basis defined by the embedding of the root lattice in the sectors which contain the gauge bosons. The 46 gauge bosons of $SO(10)\times U(1)$ are distributed in 8 sectors as shown in Table 1.

In the untwisted sector, massless gauge bosons arise from states with a spacetime fermion excited and a pair of left-moving Weyl (or pseudo-complex[9]) fermion modes excited. In the first 12 left-mover slots which embed $SO(10)\times U(1)$, there are 66 such pairs, but only six of these survive the projections. These six gauge bosons correspond to exciting the particle and antiparticle modes of each of the six Weyl fermions; the resulting fermionic charges for all six are (0,0,0,0,0,0). Obviously the six associated currents are the Cartan elements of $SO(10)\times U(1)$; because these Cartan currents are realized by fermion bilinears we can read off *any* weight of $SO(10)\times U(1)$ from the six corresponding fermionic charges.

The embedding of $SO(10)$ in these six fermionic charges is completely characterized by the fermionic charges of the five simple roots [17]:

$$
\begin{array}{cccccc}
(\ 0, & 0, & 0, & 1, & 0, & 0) \\
(1/2, & -1/2, & -1/2, & -1/2, & 0, & 0) \\
(\ 0, & 0, & 1, & 0, & 0, & 0) \\
(\ 0, & 1/2, & -1/2, & 0, & -1/2, & 1/2) \\
(\ 0, & 1/2, & -1/2, & 0, & 1/2, & -1/2)
\end{array}
$$

It is apparent then that the $U(1)$ weight is proportional to the sum of the fifth and sixth fermionic charges.

There are additional gauge bosons in the untwisted sector which arise from exciting one of the six Weyl fermions just discussed together with a mode from one of the seven pseudo-complex left-movers comprising the block of real fermions. There are 12 distinct fermionic charges which could result: $(\pm 1,0,0,0,0,0)$, $(0,\pm 1,0,0,0,0)$, etc.. However after

---

[9] See section 5.3 for a discussion of pseudo-complexification.

the GSO projections only four of these appear in gauge boson states: $(\pm1,0,0,0,0,0)$ and $(0,\pm1,0,0,0,0)$.

Let us consider the other sectors which contain gauge bosons in turn. Massless gauge bosons from V_2 arise when all the left-movers are in the vacuum state. The first 12 left-mover slots of V_2 are (111111110000); the associated fermionic charges are

$$(\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},0,0) \qquad . \tag{4.6}$$

All of these charges correspond to roots of $SO(10)$, however, only 8 of these 16 charges appear in gauge boson states after the projections. The other 8 of these 16 charges appear in the gauge boson states in V_2+V_3. Note that V_2 and V_2+V_3 differ only by the boundary conditions of the real fermions, thus it is the real fermion structure which correlates the GSO projections in these two sectors. Massless gauge bosons from V_3 require one excited left-moving Weyl (or pseudo-complex) fermion mode. The first 12 left-mover slots of V_3 are (000000000000). There are 12 possible fermionic charges for massless gauge bosons of $SO(10)\times U(1)$: $(\pm1,0,0,0,0,0)$, $(0,\pm1,0,0,0,0)$, etc.. However after the projections only four of these appear in gauge boson states: $(0,0,\pm1,0,0,0)$ and $(0,0,0,\pm1,0,0)$.

Massless gauge bosons from V_4 arise when all the left-movers are in the vacuum state; the associated fermionic charges are

$$(\pm\frac{1}{2},0,0,\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2}) \qquad . \tag{4.7}$$

Now for a state to be neutral under the extra $U(1)$ of $SO(10)\times U(1)$, the sum of the 5th and 6th fermionic charges must be zero. Thus only 8 of the 16 charges in (4.7) correspond to roots of $SO(10)$. Of these 8, only four appear as gauge bosons in V_4 after the projections. The other four appear as gauge boson states in V_3+V_4. Lastly, the gauge bosons coming from V_2+V_4 and $V_2+V_3+V_4$ are exactly analogous to the above discussion of V_4 and V_3+V_4. Table 2 summarizes the fermionic charges of the 45 $SO(10)$ gauge bosons.

Thus we have understood the gauge bosons and fermionic charges corresponding to all 45 roots of $SO(10)$; this defines an explicit embedding of the gauge group into 6 fermionic charges. It is then easy to translate the weights of any other irrep into fermionic charges, and thus read off the gauge quantum numbers for all the massless states in the spectrum. Of course, because of the $N=1$ spacetime supersymmetry, the massless matter fields group into chiral supermultiplets containing a complex scalar and a Weyl spinor. Because the gravitino resides in sector V_1, the superpartner of a boson/fermion in sector $\alpha_i V_i$ must

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always be in sector $V_1 + \alpha_i V_i$. It is a convenient shorthand when we count "states" in the massless spectrum to count them four at a time: two scalars and two CPT conjugate spinor states.

In this model the embedding of $SO(10) \times U(1)$ is such that fermionic charges $(1/2,1/2,0,0,1/4,-1/4)$ indicate the highest weight of a 16 of $SO(10)$, with $U(1)$ charge zero. It is obvious, therefore, that this model contains no neutral 16's, since these require boundary conditions $(++--)$ in left-mover slots 9 through 12. On the other hand, fermionic charges $(1/2,1/2,0,0,1/2,0)$ indicate the highest weight of a 16 of $SO(10)$, with $U(1)$ charge 1/2. Examining the basis vectors we immediately see that sectors V_5, V_6, and V_7 all potentially contribute states of a massless 16. After performing the projections one finds that in fact V_5 and V_6 contribute the highest weights of two chiral 16's each. However V_7 does not contribute any massless states at all to the spectrum: the projection from V_9 removes them. This feature is independent of the choice of k_{ij}'s; it depends only on the overlap between V_7 and V_9.

The 16's are chiral because the helicity is correlated with the $SO(10)$ weight which distinguishes the 16 from the $\overline{16}$. One also finds that sector $V_6 + V_8$ contributes the highest weights of two $\overline{16}$'s; these may couple via adjoint Higgs in sector V_8 to the two 16's in V_6, making them superheavy.

It is useful to observe that if the highest weight state of a 16 resides in, say, sector V_5, then the states which fill out this irrep must reside either in V_5 or in sectors which are the sum of V_5 and a sector containing $SO(10)$ gauge bosons. Thus, e.g., for either of the two 16's whose highest weight is in V_5, the full irrep consists of four states from V_5 and two states each from $V_2 + V_5$, $V_4 + V_5$, $V_2 + V_3 + V_5$, $V_2 + V_4 + V_5$, $V_3 + V_4 + V_5$, and $V_2 + V_3 + V_4 + V_5$. Note that no states of the 16 come from $V_3 + V_5$ in this example, but in general some could.

The full gauge group of this model is $SO(10) \times SO(8) \times [U(1)]^4$. $SO(8)$ is a hidden sector gauge group and is realized at level one. However the embedding of $SO(8)$ is nontrivial: the 28 gauge bosons are distributed in the 16 different sectors which can be formed from linear combinations of V_2, V_3, V_4, and $2*V_9$. Hidden sector massless fields occur in the singlet, 8_v, 8_s, and 8_c irreps of $SO(8)$. The full massless spectrum of chiral superfields for Model A is listed in Table 3. The $U(1)$'s associated with the first two charges listed are anomalous; the linear combination $2 \cdot Q_1 + Q_2$ is anomaly-free.

The role of the block of 16 real fermions in this model is twofold. First it reduces the rank of the gauge group. The maximal rank for the gauge group from the left-movers is

22; this is reduced by nine because of the three Ising fermions and the 15 left-moving real fermions. Thus the full gauge group has rank 13.

The second role of the real fermions is that they make it possible to embed a higher level current algebra, simultaneously producing a discrete holomorphic algebra. From the discussion above of the gauge bosons it is easy to deduce how this model realizes the 45 currents of $SO(10)$ at level two. The Cartan elements, as already mentioned, are fermion bilinears of the form $\lambda^\dagger \lambda$ and don't involve the real fermions. There are four other currents which are also fermion bilinears, but where one of the fermions is pseudo-complex. From V_3 we see that there are four currents which are composites of one Weyl fermion with 8 real fermion twist fields. Lastly, there are 32 currents which are composites of 4 Weyl fermion twist fields with 8 real fermion twist fields.

To see the importance of the discrete holomorphic operator algebra, consider the massless adjoint Higgs in this model. There are two 45 Higgs supermultiplets in Model A; the scalars are distributed in sectors as shown in Table 4.

Unlike the gauge bosons, these adjoint Higgs are not associated with the $SO(10)$ currents, rather they correspond to *primary* fields with respect to the level two $SO(10)$ Kac-Moody current algebra. These holomorphic primaries have conformal dimension 4/5. Since the operators which create physical states must have left conformal dimension 1, the adjoint Higgs must be a nontrivial element of the discrete operator algebra. This is encoded in the real fermion structure of V_8.

It is interesting to note that even after fixing the embedding of $SO(10)$ in fermionic charges, there is still some residual freedom to adjust the accompanying real fermion structures. This can be seen by comparing Model A with the $SO(10)$ level two model of Lewellen [17]. Lewellen's model can be obtained from Model A by replacing V_5–V_9 with the following:

V_5: $(11100100010010010010\|111100001100|1010101010101010|1100000000000000)$
V_6: $(11010010100100001001\|000000001111|0000000000000000|0011110000000000)$
V_7: $(00000000101101101101\|111111110000|0000000000000000|0000000000000000)$

The k_{ij} for $i>j$ and k_{00} are all zero in this model.

Lewellen's model embeds $SO(10) \times U(1)$ into six fermionic charges in exactly the same way as Model A. However the real fermion content of the $SO(10)$ currents is slightly different. In particular, for Lewellen's model the untwisted sector contributes only the six

Cartan gauge bosons, while V_3 contributes eight gauge bosons instead of four. This means that there are no currents which are fermion bilinears and where one of the fermions is pseudo-complex; it also means that there are eight rather than four currents which are composites of one Weyl fermion with 8 real fermion twist fields.

Such slight differences in the real fermion structure can have important consequences for model building. For example, Model A has a more natural embedding of $SU(5) \in SO(10)$ than Lewellen's model. By simply setting $k_{93} = -1/2$, the level two $SO(10)$ of Model A is broken to a level two $SU(5)$, times a $U(1)$. This is possible because, in Model A, all of the roots of $SO(10)$ which are *not* also roots of $SU(5) \times U(1)$ are realized as gauge bosons in sectors involving V_3. Modifying k_{93} causes these gauge bosons to be projected out. Notice that the central charge of $SU(5)$ at level 2, $c = 48/7$, is not half-integer valued. Neither is that of the discrete holomorphic algebra, which has $c = 12 - (48/7)$.

4.2. Model B

This example has $N = 2$ spacetime supersymmetry, $SO(10)$ realized at level two, and Higgs in the 54 of $SO(10)$. As in Model A, the five Cartan currents are realized as simple fermion bilinears in the untwisted sector. However in Model B these currents are linear combinations of fermion bilinears corresponding to 10 left-moving Weyl fermions. The roots of $SO(10)$ are embedded in 10 fermionic charges, corresponding to the first 20 left-mover slots. The next 16 left-mover slots are again a block of 16 real fermions.

V_0: $(11111111111111111111\|11111111111111111111\|1111\|1111111111111111\|1111)$
V_1: $(11100100100100100100\|00000000000000000000\|0000\|0000000000000000\|0000)$
V_2: $(00000000000000000000\|11111111000000000000\|0000\|1111111100000000\|0000)$
V_3: $(00000000000000000000\|11110000111100000000\|0000\|1111000011110000\|0000)$
V_4: $(00000000000000000000\|11110000000011110000\|0000\|1111000000001111\|0000)$
V_5: $(00000000000000000000\|11110000000000001111\|0000\|0000000000000000\|0000)$
V_6: $(11100100010010010010\|-------------------\|----\|1100110011001100\|1100)$
V_7: $(00000000000000011000\|00000000000000000000\|0000\|0110011001100110\|0110)$

Model B

The k_{ij} for $i>j$ and k_{00} are all zero except for the following which are equal to $-1/2$: k_{50}, k_{52}, k_{53}, and k_{54}.

The embedding of $SO(10)$ in 10 fermionic charges is completely characterized by the fermionic charges of the five simple roots:

$$
\begin{array}{rrrrrrrrrr}
(1/2, & 1/2, & -1/2, & -1/2, & 0, & 0, & 0, & 0, & 0, & 0) \\
(0, & 0, & 1/2, & 1/2, & -1/2, & -1/2, & 0, & 0, & 0, & 0) \\
(0, & 0, & 0, & 0, & 1/2, & 1/2, & -1/2, & -1/2, & 0, & 0) \\
(0, & 0, & 0, & 0, & 0, & 0, & 1/2, & 1/2, & 1/2, & 1/2) \\
(0, & 0, & 0, & 0, & 0, & 0, & 1/2, & 1/2, & -1/2, & -1/2)
\end{array}
$$

This model has four Ising fermions; since there are also 16 real fermions the rank of the full gauge group coming from the left-moving fermions is 12. There are in addition two $U(1)$ gauge bosons which are part of the $N=2$ supergravity multiplet; these states arise in the untwisted sector from exciting a left-moving spacetime boson mode and exciting a right-moving Weyl fermion. Apart from these the full gauge group is

$$ SO(10) \times F_4 \times SO(5) \times U(1) \qquad , $$

where the hidden sector gauge group $F_4 \times SO(5)$ is realized at level one.

The left movers are again separated into four blocks: the first 20 left-mover slots denote 10 Weyl fermions whose fermionic charges embed $SO(10)$, the next 4 are two more Weyl fermions which embed the $U(1)$ and part of F_4, the next 16 are the real fermions, and the remaining 4 are Ising fermion spin structures. In this example the embeddings of the visible and hidden gauge groups overlap: $SO(10)$ is embedded in the first 10 fermionic charges; F_4 is embedded in fermionic charges 3 through 8, 11, and 12; and $SO(5)$ is embedded in fermionic charges 1, 2, 9, and 10.

For Model B the 45 gauge bosons of $SO(10)$ are distributed in 11 sectors as shown in Table 5.

The five Cartan currents are linear combinations of fermion bilinears of the form $\lambda^\dagger \lambda$. There are 36 more currents which are composites of 4 Weyl fermion twist fields with 8 real fermion twist fields. The remaining 4 currents are composites of 4 Weyl fermion twist fields with a pseudo-complex fermion from the block of 16 real fermions. These 4 currents correspond to the gauge boson states which arise in V_5. Table 6 summarizes the fermionic charges of the 45 $SO(10)$ gauge bosons.

Because of the $N{=}2$ spacetime supersymmetry, the massless spectrum assembles into $N{=}2$ supermultiplets. Apart from the supergravity multiplet, there are 2608 massless states which belong to supermultiplets containing either (i) a gauge boson, two Weyl spinors, and a complex scalar, or (ii) two Weyl spinors and two complex scalars. Thus the supermultiplets containing the 108 gauge bosons of $SO(10){\times}F_4{\times}SO(5){\times}U(1)$ account for 864 states; the remaining states form 218 matter supermultiplets in the following irreps:

— one 54 of $SO(10)$,

— one 26 of F_4,

— one 5 of $SO(5)$,

— four pairs $16+\overline{16}$ of $SO(10)$ which also carry charge $1/4$, $-1/4$ respectively under the $U(1)$,

— a pair which carry only $U(1)$ charge ±1, and three which are singlets under the full gauge group.

For $SO(10)$ at level two, the 54 and the 45 are the only new irreps which can occur as massless matter states other than the irreps which also occur at level one (the singlet, 10, 16, and $\overline{16}$). As was discussed above, a 45 Higgs corresponds to a level two Kac-Moody primary with conformal dimension $4/5$, and must therefore be a nontrivial element of the discrete algebra. A 54 Higgs corresponds to a level two Kac-Moody primary with conformal dimension 1; since the full physical vertex operator also has left conformal dimension 1, this implies that it must be the identity element under the discrete algebra. It is not surprising then that the states of the 54 arise in precisely the same sectors as the $SO(10)$ gauge bosons, which are also trivial under the discrete algebra. Moreover, if we construct Table 7 listing the sectors and fermionic charges of the (scalar) states in the 54, it differs from Table 6 *only* by the states in the untwisted sector.

The highest weight states of the (nonchiral) 16's arise in sectors $3{*}6$ or $3{*}6{+}7$, reflecting that fact that with this embedding of $SO(10)$ the highest weight of the 16 is given by

$$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) \qquad .$$

There are many variations of Model B which preserve the realization of $SO(10)$ at level two. For example, we can add the following additional basis vector:

V_8: $(11001001001001100100\|00000000000011110000|1111|0000000000000000|0000)$

The additional k_{ij} for $i>j$ are chosen to be all zero except for $k_{84}=-1/2$, and $k_{86}=1/4$.

For this model the $N=2$ spacetime supersymmetry of Model B is broken to $N=1$. The full gauge group is given by

$$SO(10) \times Sp(6) \times SO(5) \times SU(2) \times U(1) \qquad ,$$

which is again rank 12. The $SO(10)$ is realized at level two, and the other factors at level one.

In closing this section on examples we should emphasize that our symbolic manipulation package makes the construction and analysis of such models quite easy. All of the results presented here come directly from the computer printout, and were produced in approximately one minute on a *NeXT*. Anyone who has gained some familiarity with the modular invariance constraints could produce and analyze dozens of variations on Models A and B in a single afternoon.

5. Aspects of real fermionization

5.1. Tree-level Couplings

The tree-level correlation functions of the $N=(2,0)$ superconformal field theory are an essential ingredient in extracting the full tree-level superpotential of the low-energy effective field theory. Any solution to string theory that realizes a higher level current algebra must, if it has a fermionic embedding, *necessarily* contain some number of real fermion constituents, i.e., Majorana-Weyl fermions which cannot be paired into either Ising or Weyl fermions in every sector of the partition function. The correlators of a real fermion conformal field theory cannot be abstracted from those of the critical Ising model or of free bosons, and thus require an independent analysis.

In the fermionic construction given by Kawai, Lewellen, Schwartz, and Tye (KLST), any three sectors of the partition function allow a *pseudo-complexification*: a pairing of the real fermions that is consistent with their boundary conditions in each of the three sectors [16]. This property of their construction is motivated by requiring modular invariance of non-vanishing two loop amplitudes in the factorization limit. Conservation of the pseudo-U(1) charges associated with such pseudo-complexifications then provides a *prescription* for computing arbitrary 3-point and 4-point correlators involving real fermions. However even this prescription breaks down for general N-point correlators, $N>4$. Clearly, it would

be useful to have a more complete understanding of real fermion conformal field theories, both as a consistency check on the limits of the validity of the KLST prescription, and with a view towards developing direct tree-level methods that can be extended to other cases of interest.

Let us consider an alternative starting point. For rational conformal field theories, such as real fermions, Verlinde's theorem [45] allows us to make explicit contact between the modular transformation properties of the chiral spin structure blocks in the one-loop partition function, and the tree-level fusion algebra of the chiral primary field operators. The correspondence works as follows. In a rational conformal field theory it is possible to rewrite the one-loop partition function in terms of a finite number of holomorphic blocks, $\chi_i(\tau)$, which are the characters of the chiral primary fields, $\phi_i(z)$, under the Virasoro algebra (or an extension thereof). Using the characters, one can form a suitable basis for the action of the modular transformations, $S : \tau \rightarrow -1/\tau$, and $T : \tau \rightarrow \tau+1$, such that S and T are realized as finite dimensional unitary matrices. It is easy to show that if the characters are modular functions the matrices S and T satisfy two important consistency conditions:

$$(ST)^3 = S^2 = C \qquad . \tag{5.1}$$

Here C is the *conjugation* matrix that takes each character to its conjugate, and satisfies $C^2 = \mathbf{1}$, the unit matrix. The existence of a conjugation matrix is related to the fact that in the tree-level operator product algebra, every chiral primary field operator is associated with a unique conjugate: let $[\phi_i]$, $[\phi_i^c]$ denote the conformal families whose chiral primary fields are ϕ_i and ϕ_i^c, respectively, and let $[\mathcal{I}]$ denote the conformal family of the identity operator. Then

$$[\phi_i] \times [\phi_i^c] \quad = \quad [\mathcal{I}] \qquad , \tag{5.2}$$

defines the chiral primary field operator, ϕ_i^c, conjugate to ϕ_i. Of course an operator could be *self*-conjugate. Verlinde's theorem is the statement that the matrix S, derived in an appropriate basis from the characters, diagonalizes (and determines) the tree-level fusion rules. Let the subscript '0' denote the conformal family of the identity operator, $\mathcal{I}$. Note that in a unitary conformal field theory the identity is the unique operator with conformal dimension zero. Construct

$$N_{ijk} = \sum_n \frac{S_{in} S_{jn} S_{nk}}{S_{0n}} \qquad , \tag{5.3}$$

where the coefficients N_{ijk} are nonnegative integers. The fusion rules are then given by

$$[\phi_i] \times [\phi_j] = N_{ijl} C^{lk} [\phi_k] \qquad . \tag{5.4}$$

The N_{ijk} also give selection rules on the 3-point chiral correlators since

$$\langle \phi_i(z_1) \phi_j(z_2) \phi_k(z_3) \rangle \propto N_{ijk} \qquad . \tag{5.5}$$

A single left-moving Majorana-Weyl fermion corresponds to a $c_L=1/2$ conformal field theory. The Virasoro primaries have conformal dimension 0 (the identity, $\mathcal{I}$), 1/2 (the chiral fermion field $\psi(z)$), or 1/16 (the chiral twist fields). In general (see [57]) there may be two distinct chiral twist fields $\sigma(z)$ and $\mu(z)$; this is the case if we require the existence of a well-defined chiral fermion number, i.e. an operator $(-1)^{F_L}$ which anticommutes with $\psi(z)$:

$$\left\{ (-1)^{F_L}, \psi_n \right\} = 0 \tag{5.6}$$

for all modes ψ_n. Acting on the Neveu-Schwarz vacuum $|0\rangle$, $\sigma(0)$ and $\mu(0)$ create two degenerate Ramond vacua with different fermion number. The Ramond zero mode operator ψ_0, $(\psi_0)^2=1/2$, takes one Ramond vacuum into the other. This implies the obvious fusion rule

$$[\psi] \times [\sigma] = [\mu] \qquad . \tag{5.7}$$

To apply Verlinde's theorem, the chiral spin structure blocks of the one-loop partition function should be rewritten in terms of the *four* chiral Virasoro characters χ_0, χ_σ, $\chi_{1/2}$, and χ_μ. Of course the Virasoro characters $\chi_\sigma(\tau)$ and $\chi_\mu(\tau)$ are actually equal, since the corresponding primaries have the same left conformal dimension. We write [57]

$$\begin{aligned}
\mathcal{Z}_0^0(\tau) &\equiv \chi_0(\tau) + \chi_{1/2}(\tau) \\
\mathcal{Z}_1^0(\tau) &\equiv \chi_0(\tau) - \chi_{1/2}(\tau) \\
\mathcal{Z}_0^1(\tau) &\equiv \tilde{\chi}_\sigma(\tau) + \tilde{\chi}_\mu(\tau) \\
\mathcal{Z}_1^1(\tau) &\equiv \tilde{\chi}_\sigma(\tau) - \tilde{\chi}_\mu(\tau) \quad ,
\end{aligned} \tag{5.8}$$

where we have introduced the notation $\tilde{\chi}_\sigma \equiv \chi_\sigma/\sqrt{2}$, $\tilde{\chi}_\mu \equiv \chi_\mu/\sqrt{2}$. If we use the basis χ_0, χ_σ, $\chi_{1/2}$, χ_μ, to construct S, then S will not be unitary; this reflects the fact that one does not obtain a *diagonal* modular invariant using *all four* characters. We have adapted Verlinde's analysis to this case, however here we will employ the convenient shortcut of using the modified basis χ_0, $\tilde{\chi}_\sigma$, $\chi_{1/2}$, $\tilde{\chi}_\mu$.

Since the Ramond-Ramond block $\mathcal{Z}_1^1(\tau)$ vanishes, it may not seem that its modular transformation properties under S and T are meaningful. However it is apparent in the KLST formalism that $\mathcal{Z}_1^1(\tau)$ picks up phases under S and T, and that these phases are vital to the construction of the partition function for real fermions. In [16] this was understood by appealing to higher loop modular invariance: although $\mathcal{Z}_1^1(\tau)$ vanishes, it appears in the factorization limit of certain *nonvanishing* two-loop amplitudes. Here we see that the modular transformation properties of $\mathcal{Z}_1^1(\tau)$ are needed to connect the one-loop partition function to the tree-level fusion rules. Both arguments may be regarded as appealing to the unitarity of the internal rational conformal field theory. To be completely general, we will parametrize the modular transformations of $\mathcal{Z}_1^1(\tau)$ by two phases:

$$
\begin{aligned}
\tau \to -1/\tau : \quad & \mathcal{Z}_0^0 \to \mathcal{Z}_0^0 \\
& \mathcal{Z}_1^0 \to \mathcal{Z}_0^1 \\
& \mathcal{Z}_0^1 \to \mathcal{Z}_1^0 \\
& \mathcal{Z}_1^1 \to e^{i\phi}\,\mathcal{Z}_1^1 \\
\tau \to \tau + 1 : \quad & \mathcal{Z}_0^0 \to e^{-\frac{\pi i}{24}}\,\mathcal{Z}_1^0 \\
& \mathcal{Z}_1^0 \to e^{-\frac{\pi i}{24}}\,\mathcal{Z}_0^0 \\
& \mathcal{Z}_0^1 \to e^{\frac{\pi i}{12}}\,\mathcal{Z}_0^1 \\
& \mathcal{Z}_1^1 \to e^{i\eta}e^{\frac{\pi i}{12}}\,\mathcal{Z}_1^1 \quad .
\end{aligned}
\tag{5.9}
$$

The parameters ϕ and η are then fixed by combining (5.8) with (5.9) and imposing the consistency conditions (5.1). Thus requiring $(ST)^3 = S^2$ gives

$$
\eta = \frac{\pi}{12} - \frac{\phi}{3} \quad .
\tag{5.10}
$$

The constraint $S^4 = \mathbf{1}$ has *two distinct solutions*:

$$
\phi = 0, \quad \frac{\pi}{2} \quad .
$$

We thus obtain two possible forms for S acting as a 4×4 unitary matrix on the modified basis set χ_0, $\tilde{\chi}_\sigma$, $\chi_{1/2}$, and $\tilde{\chi}_\mu$:

$$
\phi = 0 : \quad S = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}
$$

$$
\phi = \frac{\pi}{2} : \quad S = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \quad .
\tag{5.11}
$$

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Verlinde's theorem then provides the corresponding tree-level fusion rules:

$$
\begin{aligned}
\phi = 0 : \quad & [\psi] \times [\psi] = [\mathcal{I}] \\
& [\psi] \times [\sigma] = [\mu] \\
& [\sigma] \times [\sigma] = [\mathcal{I}] \\
& [\mu] \times [\mu] = [\mathcal{I}] \\
& [\sigma] \times [\mu] = [\psi] \\
\phi = \frac{\pi}{2} : \quad & [\psi] \times [\psi] = [\mathcal{I}] \\
& [\psi] \times [\sigma] = [\mu] \\
& [\sigma] \times [\sigma] = [\psi] \\
& [\mu] \times [\mu] = [\psi] \\
& [\sigma] \times [\mu] = [\mathcal{I}]
\end{aligned}
\tag{5.12}
$$

We will refer to the $\phi=0$ case as the *s-type* fusion rules, for self-conjugate twist fields, and the $\phi=\pi/2$ case as the *c-type* fusion rules. In the latter fusion algebra the twist fields are conjugates of each other.

Our result is that in any solution obtained via real fermionization each constituent real fermion can be labelled as s-type or c-type, where this *labeling* denotes the corresponding set of fusion rules. It is important to realize that this should *not* be regarded as a new result in the conformal field theory of free Majorana-Weyl fermions *per se*, rather it is a new result about the proper conformal field theory interpretation of solutions to string theory obtained in the fermionic formulation.

To emphasize this last point, we sketch how to recover the familiar fusion rules of the Ising model. The critical Ising model does not require the existence of a chiral $(-1)^{F_L}$, only of the non-chiral combination $(-1)^F=(-1)^{F_L+F_R}$. Thus for the Ising model we need introduce only a single chiral twist field $\sigma^+(z)$, where $\sigma^\pm(z)=(\sigma(z)\pm\mu(z))/\sqrt{2}$. The unitary matrix S is now computed in the new basis provided by the four chiral Virasoro characters χ_0, χ_{σ^+}, $\chi_{1/2}$, and χ_{σ^-}. The result is identical for the s-type and c-type cases:

$$
S = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 & 0 \\ \sqrt{2} & 0 & -\sqrt{2} & 0 \\ 1 & -\sqrt{2} & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad .
\tag{5.13}
$$

Clearly $\sigma^-(z)$ decouples; it can be consistently set to zero. Application of Verlinde's theorem then gives the fusion rules:

$$[\psi] \times [\psi] = [\mathcal{I}]$$
$$[\psi] \times [\sigma] = [\sigma] \tag{5.14}$$
$$[\sigma] \times [\sigma] = [\mathcal{I}] + [\psi] \quad ,$$

where the superscript $+$ on σ has been dropped. These are the familiar fusion rules appearing in, e.g., [45].

5.2. Selection Rules

Given explicit fusion rules for the chiral primaries of the real fermions the correlators can be obtained via the conformal bootstrap. We intend to give a complete treatment of such computations in future work. A useful means of finding selection rules for correlators is to introduce the notion of *simple currents* (also called bonus currents), discussed for general rational conformal field theories in [58][59]. A simple current is defined as any chiral primary $\phi_i(z)$ in the chiral operator product algebra such that

$$\sum_k N_{ij}^k = 1 , \quad \text{for all } j. \tag{5.15}$$

For example, in the Ising fusion rules (5.14), $\psi(z)$ is a simple current, but $\sigma(z)$ is not.

In general simple currents are not currents, i.e. they need not have conformal dimension $=1$. However associated with each simple current is a discrete symmetry, and a corresponding charge which is conserved *mod* 1 in correlators. This is easy to demonstrate for the fusion algebras (5.12) obtained above. For any simple current $\phi_i(z)$, there must be a positive integer N such that $[(\phi_i)^N]=\mathcal{I}$. N is called the *order* of the simple current. Thus for example in the s-type algebra (5.12), $\sigma(z)$ is a simple current of order 2, while in the c-type algebra $\sigma(z)$ is a simple current of order 4. Clearly the chiral primaries of any rational fusion algebra can be decomposed into orbits with respect to each simple current. Thus in the s-type algebra, the orbits with respect to $\sigma(z)$ are $\{\mathcal{I},\sigma\}$, $\{\mu,\psi\}$; for the c-type algebra, there is only one orbit: $\{\mathcal{I},\sigma,\psi,\mu\}$.

For any simple current $\phi_i(z)$, there is a discrete charge Q_j assigned to every primary $\phi_j(z)$. When the matrix S is symmetric (as in (5.11)), these charges are given by the simple expression[59]:

$$e^{2\pi i Q_j} = \frac{S_{ij}}{S_{oj}} \quad . \tag{5.16}$$

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These charges are conserved *mod* 1 in correlators. This provides useful selection rules for N-point functions involving real fermions. One of these selection rules is already familiar: $\psi(z)$ is a simple current with an associated Z_2 charge. This charge is the same for the s and c-type algebras. Conservation of this charge gives the selection rule that correlators with an *odd* number of Ramond fields vanish [18].

5.3. Consistency of the KLST Construction

The analysis of the previous section makes an explicit connection between the one-loop partition function of real fermions, and the tree-level operator algebra of the underlying conformal field theory. This allows us to perform some consistency checks on the KLST formulation [16]. We will show that for a large class of consistent solutions, the prescription given in [16] is both necessary and sufficient. However we will also derive the simplest case where the KLST prescription appears to break down. The problem can be traced to the assumed modular tranformations of the real fermion spin structure blocks.

The KLST prescription includes three constraints which apply only to the real fermion spin structures in the partition function. These are [16]:

(i) The total number of real fermions is even.

(ii) Let $O(V_i, V_j)$ denote the number of overlaps of real fermions with the Ramond boundary condition between sectors V_i and V_j. Then for all V_i, V_j, $O(V_i, V_j)$ must be even.

(iii) Let $O(V_i, V_j, V_k)$ be the number of overlaps of real fermions with the Ramond boundary condition common to three sectors. Then for all V_i, V_j, V_k, $O(V_i, V_j, V_k)$ must be even. This is referred to as the cubic constraint in [16][18][17]. Note that, since the all-Ramond basis vector V_0 is in every model, (ii) is actually implied by (iii). By the same token, $O(V_0, V_i)$ even implies that the total number of real fermions with the Ramond boundary condition in any single basis vector must be even.

The KLST construction relies on pseudo-complexification of pairs of real fermions in order to define the Fock space upon which the GSO projection operators act. Pseudo-complexification means that, in every sector, real fermions are sorted —in a sector-dependent way— into NS-NS or R-R pairs. Each pair is then used to define a complex fermion, and the Fock space is constructed as if these complex fermions were actual Weyl fermions. The resulting Fock space is obviously a subspace of the original Fock space spanned by the real fermions.

The KLST construction also relies on the pseudo-complexification of pairs of real fermions in order to define the modular transformation properties of the chiral spin structure blocks of a single real fermion. The transformation properties were assumed to be given (up to a sign) by the "square root" of those for a Weyl fermion. Thus

$$
\begin{aligned}
\tau \to -1/\tau : \quad & \mathcal{Z}_0^0 \to \mathcal{Z}_0^0 \quad \mathcal{Z}_1^0 \to \mathcal{Z}_0^1 \\
& \mathcal{Z}_0^1 \to \mathcal{Z}_1^0 \quad \mathcal{Z}_1^1 \to e^{\frac{\pi i}{4}} \mathcal{Z}_1^1 \\
\tau \to \tau + 1 : \quad & \mathcal{Z}_0^0 \to e^{-\frac{\pi i}{24}} \mathcal{Z}_1^0 \quad \mathcal{Z}_1^0 \to e^{-\frac{\pi i}{24}} \mathcal{Z}_0^0 \\
& \mathcal{Z}_0^1 \to e^{\frac{\pi i}{12}} \mathcal{Z}_0^1 \quad \mathcal{Z}_1^1 \to e^{\frac{\pi i}{12}} \mathcal{Z}_1^1 \quad .
\end{aligned}
\tag{5.17}
$$

One immediately notes that this does not agree with the modular transformation properties of either the s-type or the c-type cases discussed above. However in a partition function of N real fermions, the modular transformations of relevance are those of the real fermion spin structure blocks *taken N at a time*. Suppose that in a particular sector of the partition function, there are N_s, N_c left-moving real fermions with Ramond boundary condition and fusion algebra of s, c type, and $\bar{N}_s$, $\bar{N}_c$ right-moving real fermions with Ramond boundary condition and fusion algebra of s, c type. According to the transformation properties under S assumed in the KLST prescription (5.17), the corresponding real fermion spin structure blocks transform by the overall phase

$$
\exp \frac{\pi i (N_s + N_c - \bar{N}_s - \bar{N}_c)}{4} \quad .
\tag{5.18}
$$

Our analysis in the previous section indicates that the overall phase should be

$$
\exp \frac{\pi i (N_c - \bar{N}_c)}{2} \quad .
\tag{5.19}
$$

Thus consistency between the two prescriptions for the modular transformation properties is achieved if and only if

$$
(N_s + \bar{N}_c) - (N_c + \bar{N}_s) = 0 \mod 8.
\tag{5.20}
$$

for every sector in the partition function. Since the chiral spin structure blocks of right-moving c-type real fermions transform like those of left-moving s-type real fermions for the purposes of this argument, we will suppress the left-right labeling and write simply

$$
N_s - N_c = 0 \mod 8.
\tag{5.21}
$$

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This is the basic identity required for agreement between the assumed modular transformation properties in the KLST prescription, and those derived from the tree-level fusion rules of the real fermion conformal field theory.

Our task now is to convert this consistency equation into a list of constraints on the *basis vectors*. i.e., the set of boundary condition vectors which span the sectors of the partition function. One obvious consequence of (5.21), given that the sector V_0 occurs in any solution, is that the total number of real fermions in the underlying conformal field theory must be even (thus reproducing (i) above). In a sector where $N_s=N_c$ (*not* merely mod 8), there are as many real fermions with Ramond boundary condition and fusion algebra of s-type as of c-type, and as many real fermions with Neveu-Schwarz boundary condition and fusion algebra of s-type as of c-type. Thus we have a collection of s-c pairs. However a Weyl fermion with periodic or antiperiodic boundary condition may also be regarded as an s-c pair of real fermions: the holomorphic operator algebra of a Weyl fermion is a subalgebra of that obtained from the tensor product of an s-type algebra and a c-type algebra, with only 4 chiral primaries instead of the possible $4\times4 = 16$. Thus in any sector where $N_s=N_c$ we can perform a sector dependent pseudo-complexification of the real fermions. This is the essence of the KLST prescription for real fermions.

Let us now suppose that the constraint (5.21) is satisfied by the set of basis vectors and derive what additional constraints may follow by requiring (5.21) for sectors which are sums of basis vectors. To do this, let $R(V_1+V_2+...+V_k)$ denote the number of real Ramonds in the sector defined by the sum of basis vectors $V_1+V_2+...+V_k$. Then one can easily verify the following identity:

$$
\begin{aligned}
R(V_1 + V_2 + ... + V_k) = \sum_i R(V_i) - 2\sum_{i<j} O(V_i, V_j) \\
+ 4\sum_{i<j<k} O(V_i, V_j, V_k) - 8\sum_{i<j<k<l} O(V_i, V_j, V_k, V_l) + \dots
\end{aligned}
\tag{5.22}
$$

Applying (5.21) and (5.22) to the sum of any two basis vectors, one finds:

$$
O_s(V_i, V_j) - O_c(V_i, V_j) = 0 \quad \mod 4,
\tag{5.23}
$$

where O_s and O_c denote the numbers of overlaps of real fermions with Ramond boundary condition and s-type or c-type fusion algebra, respectively. Since $O(V_i, V_j) = O_s(V_i, V_j)+O_c(V_i, V_j)$, (5.23) implies constraint (ii). However (5.23) is a somewhat stronger constraint than (ii).

Applying (5.21) and (5.22) to the sum of any three basis vectors, one finds:

$$O_s(V_i, V_j, V_k) - O_c(V_i, V_j, V_k) = 0 \mod 2. \tag{5.24}$$

This is obviously equivalent to the cubic constraint (iii).

Applying (5.21) and (5.22) to the sum of any four basis vectors, one finds:

$$O_s(V_i, V_j, V_k, V_l) - O_c(V_i, V_j, V_k, V_l) = 0 \mod 1. \tag{5.25}$$

However this is no constraint at all, since O_s and O_c are integers. There is therefore no "quartic constraint" for real fermions, a fact which was first obtained by KLST [16]. Similarly looking at sums of > 4 basis vectors produces no additional constraints.

5.4. Spin Structures For Real Fermions

So far we have shown that the consistency condition (5.21) suffices to derive the KLST constraints (i)-(iii) *without* making any reference to higher-loop modular invariance. To see whether (5.21) implies any *additional* requirements beyond (i)-(iii), we will consider the general form of sets of basis vectors which describe real fermion spin-structures. We will suppress the entries of a basis vector which describe Weyl or Ising fermions, writing N dimensional basis vectors, where N is the number of real fermions. We can also suppress the distinction between left-movers and right-movers for the purposes of this argument. The real fermions are of course either periodic or antiperiodic. Furthermore, the boundary conditions have been chosen such that there are no *global pairs*, i.e. no two real fermions have identically matched boundary conditions across the entire set of basis vectors. Obviously such a pair should have been regarded as a single Weyl or Ising fermion and thus (by assumption) suppressed.

We have already shown that the KLST constraints (i)-(iii) will follow provided that (5.21) is satisfied for any *basis vector*, and that (5.23) is satisfied for any two basis vectors. Thus our strategy will be to construct sets of basis vectors which describe real fermions and also satisfy constraints (i)-(iii). The set of basis vectors therefore defines a solution to string theory built consistent with the KLST prescription. We then need to show that for any such set of basis vectors, there exists at least one s-c labeling of the N real fermions such that (5.21) and (5.23) are satisfied. It follows that there is an unambiguous definition of the tree-level fusion rules for all of the real fermions. In each case where at least one s-c labeling exists, the KLST constraints (i)-(iii) are not only necessary but also sufficient.

Consider a set of M basis vectors describing the spin structure of N real fermions. We will consider these as N dimensional vectors whose entries are either 0 (denoting Neveu-Schwarz) or 1 (denoting Ramond). For simplicity we may always assume that we have a *minimal set* of basis vectors, in the sense that if any one basis vector were to be removed, at least two real fermions would become globally paired. We will not bother to write V_0, the basis vector with all real fermions in the Ramond ground state, which is always present. Applying constraints (i)-(iii), we then derive the following results:

1. For $M \leq 3$, there are no allowed sets of basis vectors which contain real fermions.

2. For $M = 4$, there is a *unique* set of basis vectors (modulo relabeling or reshuffling the basis) which contains real fermions. This unique set of four produces 16 real fermions:

V_1: (1111111100000000)

V_2: (1111000011110000)

V_3: (1100110011001100)

V_4: (1010101010101010)

The proof is as follows. In a collection of four vectors as above, each *vertical* column is a 4-digit binary number from 0000 to 1111. To avoid any global pairing, any particular 4-digit binary must appear just once or not at all. Thus the *maximum* number of real fermions which we can describe with four basis vectors is clearly 16. Now consider the column 1111 (the first column above). It is easy to see that if 1111 is present, then constraints (i)-(iii) imply that all 16 columns must be present. On the other hand, if 1111 is absent, then (i)-(iii) have no solutions. Thus 16 is also the minimum number of real fermions, and this is in fact the unique allowed spin structure.

3. There are many s-c labelings of the structure of 16 which satisfy (5.21) and (5.23). Two examples are

$$
\begin{aligned}
& scscscscscscscsc \\
& sssscccsssssscccc
\end{aligned}
\qquad . \tag{5.26}
$$

4. It is not difficult to show [60] that 16 is the minimum number of real fermions *for any M*.

5. For $M = 5$, the allowed spin structures describe either 16 or 32 real fermions. For a collection of five basis vectors, each vertical column is a 5-digit binary between 00000 and 11111. Thus 32 is the maximum number of real fermions which can be produced, and in fact this unique structure of 32 also satisfies the constraints (i)-(iii). It can be written as

V_1: (11111111000000001111111100000000)

V_2: (11110000111100001111000011110000)

V_3: (11001100110011001100110011001100)

V_4: (10101010101010101010101010101010)

V_5: (11111111111111110000000000000000)

This form makes it clear that the structure of 32 consists of two copies of the structure of 16. The fifth basis vector merely breaks the symmetry between the two blocks of 16. Thus to get an allowed s-c labeling for the structure of 32, we merely take any two of the allowed labelings for the structure of 16.

To complete the discussion of $M=5$, we note that the constraints (i)-(iii) are all *mod* 2 constraints. It follows immediately that if there is any spin structure satisfying (i)-(iii) and describing N real fermions, then there exists another allowed structure which describes $32-N$ real fermions. This second —or "complement"— structure is obtained from the first by simply removing the columns which appear in the first structure from the structure of 32 above. Thus there are also no allowed structures with $16<N<32$.

6. For $M>5$, the classification of allowed spin structures for real fermions gets more complicated. For example, for $M=6$, an exhaustive search shows that there are allowed structures for 16, 24, 28, 32, 36, 40, 48, and 64 real fermions. The structure of 64 is maximal, and may be regarded as four blocks of 16. The structures with 36, 40, and 48 real fermions are $64-N$ complements of the structures which give 28, 24, or 16 real fermions. Thus the only essentially new structures are those giving 24[10] or 28 real fermions. The structure of 24 may be thought of as two *overlapping* blocks of 16, and inherits a number of allowed s-c labelings from those of the 16. More generally, although we have not completed the classification of all allowed spin structures for $M>5$, it is clear that a large class of the allowed structures are built from the basic block of 16, and furthermore that they inherit allowed s-c labelings in an obvious way from the component blocks.

7. The structure of 28 real fermions for $M=6$ is more interesting. It can be written as

---

[10] This structure of 24 was derived and pointed out to us by Jonathan Feng, who has also found a different structure of 28 for $M=7$.

V_1: ($\boxed{1111111100000000}$ 000000101110)
V_2: ($\boxed{1111000011110000}$ 000000111001)
V_3: (11001100 $\boxed{1100111100000000}$ 0000)
V_4: (10101010 $\boxed{1010110011000000}$ 0000)
V_5: (00000000 0000 $\boxed{1111111111111}$ 1111)
V_6: (00010001 000 $\boxed{11110101010100}$ 0101)

This structure can be thought of as three overlapping blocks of 16: two of the blocks correspond to the boxes shown above. The third block of 16 consists of the entries which are in vectors V_1, V_2, V_5, V_6 and in columns {3,4,7,8,11,12,15,16,17,18,23,24,25,26,27,28}.

The overlaps of the three blocks of 16 are sufficiently complicated that it is not clear by inspection whether this structure inherits any allowed s-c labelings. However an exhaustive search of all 2^{28} possibilities shows that for this structure of 28 *there are no s-c labelings satisfying* (5.21). Thus in this case the KLST prescription may break down: the assumed modular properties (5.18) do not agree with (5.19). This does not necessarily mean that there are no consistent solutions to string theory with this real fermion spin structure, but that one may have to go beyond the KLST construction to derive them.

Our final result is that the original KLST construction is consistent for a large class of spin structures which describe real fermions, but may fail in other cases. Just as importantly, we have also learned that the allowed spin structures for real fermions are quite restricted. This is not surprising from the point of view of rational conformal field theory, but it has important consequences for model building.

6. Conclusions

Our work suggests a number of technical issues involving real fermionization that need further analysis. It also suggests some valuable model building strategies that may enable us to eventually go beyond free fermionization. Let us begin with two technical issues which we have not yet touched on.

1. *Supercurrent constraints.* Given a better understanding of the real fermion conformal field theories it is useful to state more precisely the world-sheet supersymmetry constraints necessary for obtaining Lorentz invariance and N=1 spacetime supersymmetry. The supercurrent of the $(1,0)$ internal superconformal field theory of central charge c=9 takes the triplet form [12][13],

$$T_F(\bar{z}) \;=\; i \sum_{k=1}^{6} \psi_{3k}\psi_{3k+1}\psi_{3k+2} \qquad , \tag{6.1}$$

38

where the $\psi_i(\bar{z})$, $i=3,\ldots20$, are right-moving Majorana-Weyl fermions, grouped into six triplets.

Following [12] we will consistently choose the internal conformal field theory part of the spacetime supersymmetry currents to be embedded in the tensor product of the six individual Ramond ground states associated with ψ_3, ψ_6, ψ_9, ψ_{12}, ψ_{15}, and ψ_{18}. The related $U(1)$ current is the fermion bilinear

$$j(\bar{z}) \;\; = \;\; i\psi_3\psi_6 + i\psi_9\psi_{12} + i\psi_{15}\psi_{18} \quad , \tag{6.2}$$

which generates a $(2,0)$ extension of the world-sheet superconformal algebra [1]. Thus, the supercurrent (6.1) can be split into T_F^+ and T_F^- as follows:

$$
\begin{aligned}
T_F^\pm(\bar{z}) \;\; = \;\; & \frac{1}{\sqrt{2}} \sum_{k=1}^{3} i \left[\psi_{6k-3}\psi_{6k-2}\psi_{6k-1} + \psi_{6k}\psi_{6k+1}\psi_{6k+2} \right] \\
& \pm \left[\psi_{6k-2}\psi_{6k-1}\psi_{6k} - \psi_{6k-3}\psi_{6k+1}\psi_{6k+2} \right] \quad .
\end{aligned}
\tag{6.3}
$$

The $U(1)$ current algebra is an independent constraint on the Hilbert space of a consistent solution to string theory beyond the constraints from $(1,0)$ world-sheet supersymmetry alone. Thus the superconformal constraints on the basis vectors in a model with spacetime supersymmetry are

$$
\begin{aligned}
r_{6k-3} + r_{6k-2} + r_{6k-1} = r_{6k} + r_{6k+1} + r_{6k+2} = r_{6k-2} + r_{6k-1} + r_{6k} \\
= r_{6k-3} + r_{6k+1} + r_{6k+2} \;\; = \;\; r_1 = r_2 \quad \text{mod } 1 \quad \text{for } k = 1,2,3 \quad .
\end{aligned}
\tag{6.4}
$$

Here, r_i denote the i'th right-moving component of any basis vector. This is not the usual form of the triplet constraint stated in the literature [12], but it is equivalent in any modular invariant spacetime supersymmetric model.

If we restrict ourselves to antiperiodic and periodic boundary conditions alone for the right-moving fermions, the superconformal conditions (6.4) are sufficient to guarantee a consistent solution to string theory, assuming that the spectrum also satisfies the modular invariance constraints. We have seen in the previous section that this requires, in addition to (6.4), that we clearly identify every right-moving Majorana-Weyl fermion as either being globally paired with a right/left-moving Majorana-Weyl fermion to form a Weyl/Ising fermion, or as a member of a *valid spin structure block* of unpaired (right-moving and/or left-moving) real fermions. For this class of solutions, we now have an unambiguous prescription to build fully consistent solutions to string theory whose underlying conformal

field theory description includes *both* unpaired and paired Majorana-Weyl fermions. The two examples given in section 4 were particularly simple examples of this class, since all of the real fermions were left-movers. We will develop this class of solutions in future work. In particular, it is possible to systematically explore the options for obtaining three generations compatible with the gauge symmetry being realized at higher level.

It is more difficult to implement the supercurrent constraints for general models containing a combination of Weyl, Ising, and real fermions. This is because we have the possibility of introducing twisted boundary conditions other than periodic or antiperiodic for some of the right-moving Weyl fermions. In this case the supercurrent constraints require that, *up to an overall basis change* of the right-moving fermions, the boundary conditions in the basis vectors $\{V_i\}$ describe a set of commuting automorphisms/antiautomorphisms of the supercurrent [13][12]. A detailed discussion with many examples is given in [61]. An explicit prescription analogous to (6.4) for determining whether a *given* set of boundary conditions is valid has not been derived, and thus this class of solutions will require further analysis.[11]

2. *Verification.* As noted, we have developed a symbolic manipulation package [44] to analyze models constructed using real fermionization. The program constructs the massless physical spectrum explicitly, by solving, for every sector, the constraint equations which implement the GSO projections. The algorithm for solving these equations is fairly involved due to the complicated form of the GSO projection operators for real fermions [16], which include products of pseudo-complexified Ramond zero mode operators.

The results so obtained are of little use unless we can also develop some convincing means for verification — both of the computer program and of the detailed algorithms which the program implements. Fortunately there are some powerful overall physics consistency checks at our disposal. For example, neither the program nor the underlying algorithm "knows" about spacetime supersymmetry or gauge invariance. Thus a strong physics consistency check is to verify that all of the derived states in the massless spectrum assemble into appropriate supermultiplets and gauge multiplets.

However we want to stress that no amount of checking of *a single model* will ever be sufficient for verification of the results. It is essential, in addition, to run dozens (or hundreds) of test models with the same program, purposely attempting to generate "peculiar"

---

[11] In particular, we believe that world-sheet supersymmetry is violated for the three generation model presented in [14].

results which signal either bugs in the code or problems with the algorithm. These test models utilize spin structures that correspond to convoluted fermionic realizations of various gauge groups and/or extended spacetime supersymmetry. These solutions may not be of direct physical interest but are absolutely essential for gaining confidence in our detailed implementation of string consistency. Verification thus becomes the most time-consuming aspect of building models with free fermionization.

Free fermionization is a useful paradigm for understanding how a successful string unification model *might* work. There are valuable lessons to be gained from an in-depth understanding of this very basic tool in string theory. Of course free fermionization has its limitations. The restriction to constructing solutions which realize only those gauge groups and representations that have a fermionic embedding implies that one must be careful in interpreting the results. It is essential to have the freedom to vary the underlying constituent conformal field theories in order to avoid concluding that a desired phenomenological outcome is "impossible".

On the other hand, real fermionization allows us to sample many interesting solutions to string theory in a calculable framework. Realizing the world-sheet operator algebras in simpler constituents such as free fields provides important technical advantages. Rather than imposing modular invariance directly on the tensor product of characters under the necessary operator algebras, such as current or coset algebras, we implement the much simpler task of imposing modular invariance on the tensor product of Virasoro characters of the constituents. Furthermore, since the emission vertices of spacetime fields are realized in the primary fields of the constituent conformal field theories, their correlation functions – which define the couplings in the superpotential – are given by the tensor product of constituent conformal field theory correlators.

Since our interest is *not* in exhaustively classifying solutions to string theory but rather in identifying solutions which offer new physical insight, this repackaging of the problem will give us the capability to efficiently access phenomenologically distinct solutions. Already we can make a number of intriguing observations about phenomenological properties of real fermionization. We have identified a large number of new embeddings of GUT groups and the standard model group, realized at higher level. The two examples presented here demonstrate that different choices of embeddings lead to quite different particle content in the effective field theory. We find that a limited number of adjoint scalars, other large Higgs irreps, and exotics can appear in our models, with highly model dependent

couplings. The number of gauge-singlet moduli can also be quite small, a result which may have important phenomenological consequences. There are interesting new possibilities for the hidden sector gauge group and matter content. Last but not least, real fermionization clearly restricts the operators that give fermions mass in ways that differ strongly from previous constructions.

It might seem that, given a sufficiently wide range of constituent conformal field theories, anything and everything is possible in the spectrum and in the superpotential. This is a misconception. As we have repeatedly emphasized, and as is evident in any experience with building explicit solutions, string consistency is a very restrictive principle. Slight changes in the underlying conformal field theory embeddings can have rather drastic consequences for the massless spectrum and the superpotential. Given the dictionary between spacetime symmetries and world sheet operator algebras, it is probably not difficult to construct conformal field theory structures that realize any *single* phenomenological feature, assuming it satisfies the bounds on allowed conformal dimension and total conformal anomaly [62][28]. But the final step of piecing together *many* features in a consistent solution is extremely delicate. It is this property which makes superstring unification so restrictive, but also compelling.

References

[1] T. Banks, L. Dixon, D. Friedan and E. Martinec, Nucl. Phys. **B299** (1988) 613, and references therein.

[2] M. Dine and N. Seiberg, Phys. Lett. **162B** (1985) 65; Nucl. Phys. **B301** (1988) 357.

[3] T. Yoneya, *Prog. Theo. Phys.* **51** (1974) 1907; J. Scherk and J. H. Schwarz, Nucl. Phys. **B81** (1974) 118.

[4] M. B. Green and J. H. Schwarz, Phys. Lett. **136B** (1984) 367; Phys. Lett. **149B** (1984) 117.

[5] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory*, Cambridge, Cambridge (1987). For a recent review of conformal field theory and string theory, see J. Polchinski, *What is String Theory?*, Lectures in Les Houches LXII, hep-th/9411028 (1994).

[6] P. Candelas, G. Horowitz, A. Strominger and E. Witten, Nucl. Phys. **B258** (1985) 46. For an introduction, see E. Witten, *Grand Unification in Four or Ten Dimensions*, Sixth Workshop on Grand Unification, ed. S. Rudaz and T. Walsh (1985).

[7] D. Gross, J. Harvey, E. Martinec and R. Rohm, Nucl. Phys. **B256** (1985) 253.

[8] For an introduction, see L. Dixon, *Some world-sheet properties of superstring compactifications*, in the ICTP Summer Workshop, Trieste (1987) QCD161:W626:1987; *ibid.*, *Supersymmetry breaking in string theory*, in the DPF meeting, Houston TX (1990) QCD161:T45:1989.

[9] L. Dixon, J. Harvey, C. Vafa, and E. Witten, Nucl. Phys. **B261** (1985) 678; Nucl. Phys. **B274** (1986) 285.

[10] K. Narain, M. Sarmadi and C. Vafa, Nucl. Phys. **B288** (1987) 551.

[11] L. Ibanez, J. Mas, H-P Nilles, and F. Quevedo, Nucl. Phys. **B301** (1988) 157.

[12] H. Kawai, D. C. Lewellen, and S.-H. H. Tye, Nucl. Phys. **B288** (1987) 1.

[13] I. Antoniadis, C. P. Bachas, and C. Kounnas, Nucl. Phys. **B289** (1987) 87; I. Antoniadis and C. P. Bachas, Nucl. Phys. **B298** (1988) 586.

[14] S. Chaudhuri, S. Chung and J. Lykken, *Fermion masses from superstring models with adjoint scalars*, hep-ph/9405374, May 1994; S. Chaudhuri, S. Chung, G. Hockney and J. Lykken, Proc. of the DPF meeting, Albuquerque, NM, hep-th/9409151, August 1994.

[15] S. Kalara, J. Lopez and D. Nanopoulos, Nucl. Phys. **B353** (1991) 650.

[16] H. Kawai, D. C. Lewellen, J. A. Schwartz, and S.-H. H. Tye, Nucl. Phys. **B299** (1988) 431.

[17] D. Lewellen, Nucl. Phys. **B337** (1990) 61.

[18] J. A. Schwartz, Phys. Rev. **D42** (1990) 1777.

[19] G. Cleaver, *GUTs with adjoint Higgs fields from superstrings*, Ohio State preprint OHSTPY-HEP-T-94-007, hep-th/9409096.

[20] G. Aldazabal, A. Font, L. Ibanez, A. Uranga, *String GUTS*, FTUAM-94-28, hep-th@xxx.lanl.gov-9410206; L. Ibanez, in Susy'94, Ann Arbor (May 1994).

[21] E. Witten, Nucl. Phys. **B188** (1981) 513; S. Dimopoulos and H. Georgi, Nucl. Phys. **B193** (1982) 475; N. Sakai, Z. Phys. C11 (1982) 153. For an introduction, see G. Ross, *Grand Unified Theories*, Benjamin, New York, 1984.

[22] L. Alvarez-Gaume, M. Claudson and M. Wise, Nucl. Phys. **B207** (1982) 16; J. Ellis, L. Ibanez, and G. Ross, Phys. Lett. **113B** (1982) 283; L. Alvarez-Gaume, J. Polchinski and M. Wise, Nucl. Phys. **B221** (1983) 495; L. Ibanez, Nucl. Phys. **B218** (1983) 514; J. Ellis, J. Hagelin, D. Nanopoulos, and K. Tamvakis, Phys. Lett. **125B** (1983) 275.

[23] J. Ellis and M. Gaillard, Phys. Lett. **88B** (1979) 315; S. Dimopoulos, Phys. Lett. **129B** (1983) 417; D. Nanopoulos and M. Srednicki, Phys. Lett. **124B** (1983) 37; J. Bagger, S. Dimopoulos, H. Georgi, and S. Raby, Proc. of the Fifth Workshop on Grand Unification, Providence, Rhode Island (1984).

[24] G. Anderson, S. Dimopoulos, L. Hall, S. Raby, and G. Starkman, Phys. Rev. **D49** (1994) 3660; S. Raby, *SO(10) SUSY GUTs and fermion masses* Ohio State preprint OHSTPY-HEP-T-94-010.

[25] S. Dimopoulos and F. Wilczek, Proc. of Erice Summer School (1981).

[26] K. S. Babu and S. M. Barr, Phys. Rev. **D51** (1995) 2463; Phys. Rev. **D49** (1994) 2156.

[27] See, for example, M. Dine, P. Huet and N. Seiberg, Nucl. Phys. **B322** (1989) 301; D. Bailin, D. Dunbar and A. Love, Phys. Lett. **219B** (1989) 76; J. Casas and C. Munoz, Nucl. Phys. **B332** (1990) 189; L. Ibanez and G. Ross, Nucl. Phys. **B368** (1992) 3.

[28] A. Font, L. Ibanez, and F. Quevedo, Nucl. Phys. **B345** (1990) 389.

[29] M. Leurer, Y. Nir, and N. Seiberg, Nucl. Phys. **B398** (1993) 319; L. Ibanez and G. Ross, Phys. Lett. **332B** (1994) 100; D. Kaplan and M. Schmaltz, Phys. Rev. **D49** (1994) 3741; P. Binetruy and P. Ramond, *Yukawa textures and anomalies*, Inst. for Fundamental Theory preprint UFIFT-HEP-94-19.

[30] K. S. Babu and R. N. Mohapatra, Phys. Rev. Lett. **74** (1995) 2418.

[31] P. Ramond, Phys. Rev. **D3** (1971) 2415; A. Neveu and J. H. Schwarz, Nucl. Phys. **B31** (1971) 86; K. Bardakci and M. Halpern, Phys. Rev. **D3** (1971) 2493.

[32] F. Gliozzi, J. Scherk, and D. Olive, Phys. Lett. **65B** (1976) 282; Nucl. Phys. **B122** (1977) 253.

[33] N. Seiberg and E. Witten, Nucl. Phys. **B276** (1986) 272.

[34] L. Alvarez-Gaume, P. Ginsparg, G. Moore, and C. Vafa, Phys. Lett. **171B** (1986) 155.

[35] H. Kawai, D. Lewellen, and S.-H. H. Tye, Phys. Rev. **D34** (1986) 3794.

[36] E. Witten, Comm. Math. Phys. **92** (1984) 455.

[37] P. Goddard, W. Nahm and D. Olive, Phys. Lett. **160B** (1985) 111.

[38] S. Chaudhuri and J. Schwartz, Phys. Lett. **219B** (1989) 291.

[39] J. Lopez, D. Nanopoulos and K. Yuan, Phys. Rev. **D50** (1994) 4060.

[40] I. Antoniadis, J. Ellis, J. Hagelin, and D. Nanopoulos, Phys. Lett. **208B** (1988) 209; See also J. Lopez, D. Nanopoulos, and A. Zichichi, hep-ph/9502414.

[41] J. Lopez, D. Nanopoulos and K. Yuan, Nucl. Phys. **B399** (1993) 654.

[42] A. Faraggi, Phys. Lett. **274B** (1992) 47; Phys. Lett. **278B** (1992) 131; Phys. Lett. **302B** (1993) 202; Phys. Lett. **326B** (1994) 62; A. Faraggi, D. Nanopoulos and K. Yuan, Nucl. Phys. **B335** (1990) 347.

[43] P. Goddard, W. Nahm, D. Olive, and A. Schwimmer, Comm. Math. Phys. **107** (1986) 179; D. Bernard and J. Thierry-Mieg, Comm. Math. Phys. **111** (1987) 181; for an introduction to free fermions and current algebras, see T. Banks, *Lectures on conformal field theory*, Proceedings of the 1987 TASI school, R. Slansky and G. West eds., World Scientific, Singapore, 1988.

[44] The symbolic manipulation package will be described and documented in a future publication. Additional information is available on the World-Wide Web at http://www-theory.fnal.gov/superstrings/superstrings.html

[45] E. Verlinde, Nucl. Phys. **B300** [FS22] (1988) 360.

[46] T. Banks and L. Dixon, Nucl. Phys. **B307** (1988) 93.

[47] V. G. Kac, *Funct. Anal. App.***1** (1967) 328; R. V. Moody, *Bull. Am. Math. Soc.***73** (1967) 217.

[48] P. Goddard and D. Olive, Int. Jour. Mod. Phys. **A1**, (1986) 303.

[49] W. McKay and J. Patera, *Tables of dimensions, indices and branching rules for representations of simple algebras* (Dekker, New York, 1981).

[50] W. Lerche, A. Schellekens and N. Warner, Phys. Rep. 177 (1989) 1.

[51] Z. Horvath and L. Palla, Phys. Lett. **229B** (1989) 368.

[52] D. Friedan, Z. Qiu, and S. Shenker, Phys. Rev. Lett. **52** (1984) 1575.

[53] L. Dixon, V. Kaplunovsky and C. Vafa, Nucl. Phys. **B294** (1987) 43.

[54] R. Slansky, Phys. Rep. 79 (1981) 1.

[55] K. S. Narain, Phys. Lett. **169B** (1986) 41; K. S. Narain, M. H. Sarmadi and E. Witten, Nucl. Phys. **B279** (1987) 369.

[56] J. Distler and S. Kachru, Nucl. Phys. **B413** (1994) 213.

[57] P. Ginsparg, *Applied Conformal Field Theory*, Les Houches lectures (1988).

[58] K. Intrilligator, Nucl. Phys. **B332** (1990) 541.

[59] A. Schellekens and S. Yankielowicz, Nucl. Phys. **B327** (1989) 673; *ibid.* Int. Jour. Mod. Phys. A5, (1990) 2903.

[60] Jonathan Feng, private communication.

[61] H. Dreiner, J. Lopez, D. Nanopoulos and D. Reiss, Nucl. Phys. **B320** (1989) 401.

[62] L. Dixon, V. Kaplunovsky and J. Louis, Nucl. Phys. **B355** (1991) 649; Nucl. Phys. **B329** (1990) 27; I. Antoniadis, K. Narain, and T. Taylor, Phys. Lett. **267B** (1991)

37; V. Kaplunovsky and J. Louis, Nucl. Phys. **B422** (1994) 57; J. Casas, Z. Lalak, C. Munoz and G. Ross, Nucl. Phys. **B347** (1993) 22; T. Banks, D. Kaplan, and A. Nelson, Phys. Rev. **D49** (1994) 779; B. de Carlos, J. Casas, F. Quevedo and E. Roulet, Phys. Lett. **318B** (1993) 447.

TABLE CAPTIONS

Table 1: The eight sectors which contribute the 46 gauge bosons of $SO(10) \times U(1)$ in Model A.

Table 2: The fermionic charges of the 46 gauge bosons of $SO(10) \times U(1)$ in Model A, listed according to the sectors that they appear in.

Table 3: The complete massless spectrum of chiral superfields for Model A. A $\pm$ indicates two distinct irreps with opposite charge: thus, for example, there are a total of four 16's of $SO(10)$ and a total of twelve 10's of $SO(10)$.

Table 4: The eight sectors which contribute the 45 scalars of the adjoint Higgs in Model A.

Table 5: The eleven sectors which contribute the 45 gauge bosons of $SO(10)$ in Model B.

Table 6: The fermionic charges of the 45 gauge bosons of $SO(10)$ in Model B, listed according to the sectors that they appear in.

Table 7: The fermionic charges of the scalars of the 54 of $SO(10)$ in Model B. The ellipsis indicates that the remaining entries are identical to the those in Table 6.

Sector	No. of gauge boson states	Real fermion b.c.'s
untwisted	10	(0000000000000000)
V_2	8	(1111111100000000)
V_3	4	(0000111111110000)
V_4	4	(1100110011001100)
V_2+V_3	8	(1111000011110000)
V_2+V_4	4	(0011001111001100)
V_3+V_4	4	(1100001100111100)
$V_2+V_3+V_4$	4	(0011110000111100)

Table 1

Sector	Fermionic charges	
untwisted:	$5\times(0,0,0,0,0,0)$	
	$\pm(1,0,0,0,0,0)$	$\pm(0,1,0,0,0,0)$
V_2 :	$\pm(1/2,-1/2,1/2,-1/2,0,0)$	$\pm(1/2,-1/2,-1/2,1/2,0,0)$
	$\pm(1/2,1/2,1/2,-1/2,0,0)$	$\pm(1/2,1/2,-1/2,1/2,0,0)$
V_3 :	$\pm(0,0,1,0,0,0)$	$\pm(0,0,0,1,0,0)$
V_4 :	$\pm(1/2,0,0,1/2,1/2,-1/2)$	$\pm(1/2,0,0,-1/2,-1/2,1/2)$
V_2+V_3 :	$\pm(1/2,-1/2,-1/2,-1/2,0,0)$	$\pm(1/2,-1/2,1/2,1/2,0,0)$
	$\pm(1/2,1/2,1/2,1/2,0,0)$	$\pm(1/2,1/2,-1/2,-1/2,0,0)$
V_2+V_4 :	$\pm(0,1/2,1/2,0,1/2,-1/2)$	$\pm(0,1/2,-1/2,0,-1/2,1/2)$
V_3+V_4 :	$\pm(1/2,0,0,-1/2,1/2,-1/2)$	$\pm(1/2,0,0,1/2,-1/2,1/2)$
$V_2+V_3+V_4$:	$\pm(0,1/2,-1/2,0,1/2,-1/2)$	$\pm(0,1/2,1/2,0,-1/2,1/2)$

Table 2

Irrep of $SO(10){\times}SO(8)$	Multiplicity	$U(1)$ charges			
45	2	0	0	0	0
16	1	±1/2	0	-1/4	0
16	2	0	0	-1/4	0
$\overline{16}$	2	0	0	1/4	0
10	2	±1/2	1/2	0	0
10	1	±1/2	-1/2	0	0
10	2	0	±1/2	0	0
10	1	0	0±1/2		0
8_v	1	0	-1/2	-1/2	0
8_s	1	0	1/2	-1/2	0
8_c	1	0	1/2	1/2	0
1	3	±1	0	0	0
1	1	±1	-1/2	1/2	0
1	1	±1/2	1/2	1/2	0
1	1	±1/2	1/2	-1/2	0
1	2	±1/2	-1/2	1/2	0
1	2	±1/2	-1/2	-1/2	0
1	1	0	±1	0	0
1	2	0	±1/2	1/2	0
1	1	0	1/2	-1/2	±1/2
1	2	0	±1/2	-1/2	0
1	1	0	0	0	±1/2
1	7	0	0	0	0

Table 3

Sector	No. of states	Real fermion b.c.'s
V_8	9	(0101010101010101)
V_2+V_8	8	(1010101001010101)
V_3+V_8	4	(0101101010100101)
V_4+V_8	4	(1001100110011001)
$V_2+V_3+V_8$	8	(1010010110100101)
$V_2+V_4+V_8$	4	(0110011010011001)
$V_3+V_4+V_8$	4	(1001011001101001)
$V_2+V_3+V_4+V_8$	4	(0110100101101001)

Table 4

Sector	No. of gauge boson states	Real fermion b.c.'s
untwisted	5	(0000000000000000)
V_2	4	(1111111100000000)
V_3	4	(1111000011110000)
V_4	4	(1111000000001111)
V_5	4	(0000000000000000)
V_2+V_3	4	(0000111111110000)
V_2+V_4	4	(0000111100001111)
V_2+V_5	4	(1111111100000000)
V_3+V_4	4	(0000000011111111)
V_3+V_5	4	(1111000011110000)
V_4+V_5	4	(1111000000001111)

Table 5

Sector	Fermionic charges	
untwisted:	$5\times(0,0,0,0,0,0,0,0,0,0)$	
V_2 :	$\pm(1/2,1/2,1/2,1/2,0,0,0,0,0,0)$	$\pm(1/2,1/2,-1/2,-1/2,0,0,0,0,0,0)$
V_3 :	$\pm(1/2,1/2,0,0,1/2,1/2,0,0,0,0)$	$\pm(1/2,1/2,0,0,-1/2,-1/2,0,0,0,0)$
V_4 :	$\pm(1/2,1/2,0,0,0,0,1/2,1/2,0,0)$	$\pm(1/2,1/2,0,0,0,0,-1/2,-1/2,0,0)$
V_5 :	$\pm(1/2,1/2,0,0,0,0,0,0,1/2,1/2)$	$\pm(1/2,1/2,0,0,0,0,0,0,-1/2,-1/2)$
V_2+V_3 :	$\pm(0,0,1/2,1/2,1/2,1/2,0,0,0,0)$	$\pm(0,0,1/2,1/2,-1/2,-1/2,0,0,0,0)$
V_2+V_4 :	$\pm(0,0,1/2,1/2,0,0,1/2,1/2,0,0)$	$\pm(0,0,1/2,1/2,0,0,-1/2,-1/2,0,0)$
V_2+V_5 :	$\pm(0,0,1/2,1/2,0,0,0,0,1/2,1/2)$	$\pm(0,0,1/2,1/2,0,0,0,0,-1/2,-1/2)$
V_3+V_4 :	$\pm(0,0,0,0,1/2,1/2,1/2,1/2,0,0)$	$\pm(0,0,0,0,1/2,1/2,-1/2,-1/2,0,0)$
V_3+V_5 :	$\pm(0,0,0,0,1/2,1/2,0,0,1/2,1/2)$	$\pm(0,0,0,0,1/2,1/2,0,0,-1/2,-1/2)$
V_4+V_5 :	$\pm(0,0,0,0,0,0,1/2,1/2,1/2,1/2)$	$\pm(0,0,0,0,0,0,1/2,1/2,-1/2,-1/2)$

Table 6

Sector	Fermionic charges	
untwisted:	4×(0,0,0,0,0,0,0,0,0,0)	
	±(1,1,0,0,0,0,0,0,0,0)	±(0,0,1,1,0,0,0,0,0,0)
	±(0,0,0,0,1,1,0,0,0,0)	±(0,0,0,0,0,0,1,1,0,0)
	±(0,0,0,0,0,0,0,0,1,1)	

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Table 7